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Analysis of Crack Problems in Graded Half-space Subject to Complex Loading

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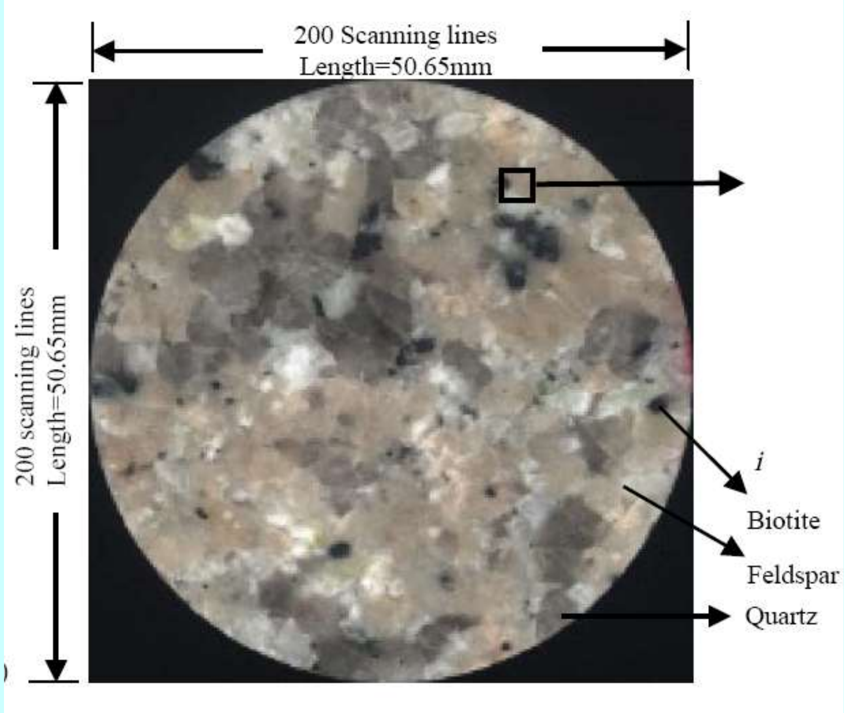
Contents

1. Introduction

The heterogeneity widely exists in natural and man-made materials. For example, granite contains three main minerals, i.e., quartz, feldspar and biotite.

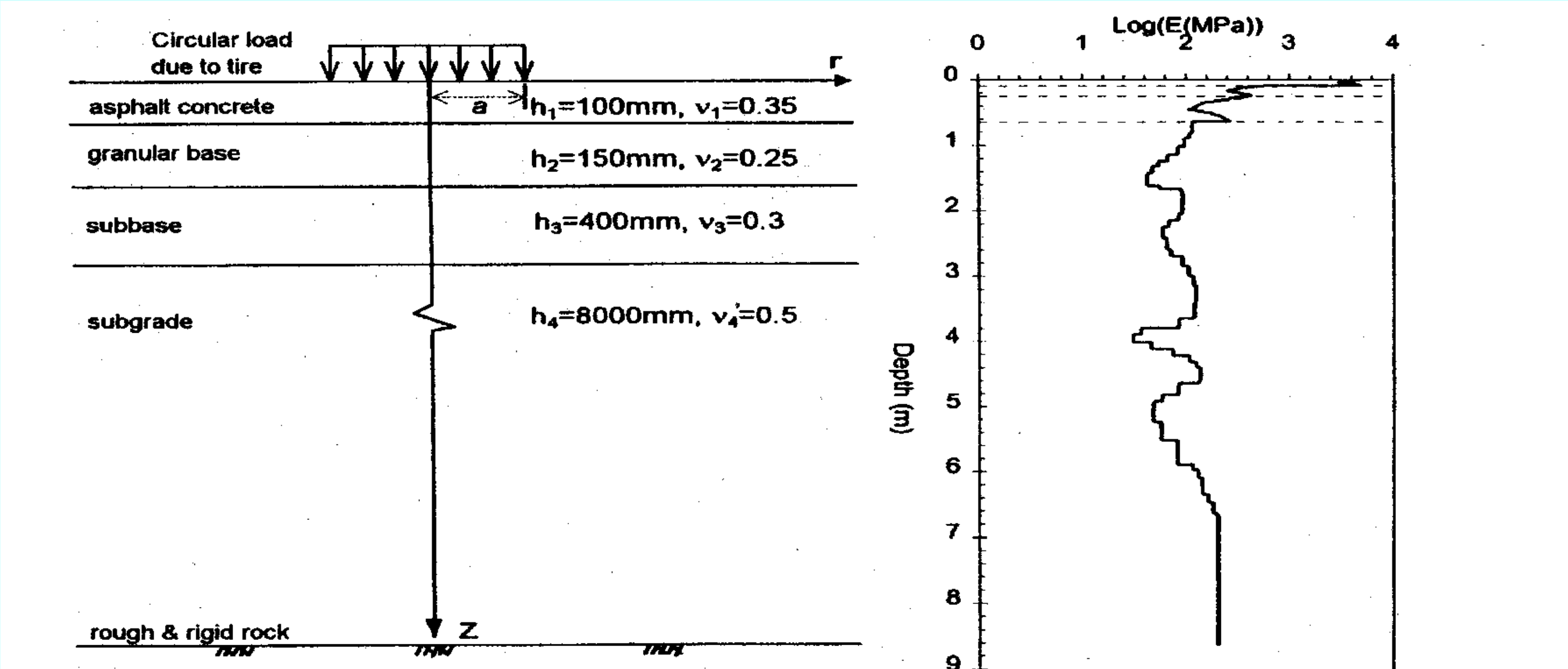


Granite image

One type of materials is characterized by the variations of the components, structures and properties of physics and mechanics along a given coordinate with only small or no variations along the other two coordinates perpendicular to it. Such materials are called functionally graded materials and can be regarded as a special type of general heterogeneous material.

Two types of non-homogeneous materials along one direction are presented in the following:

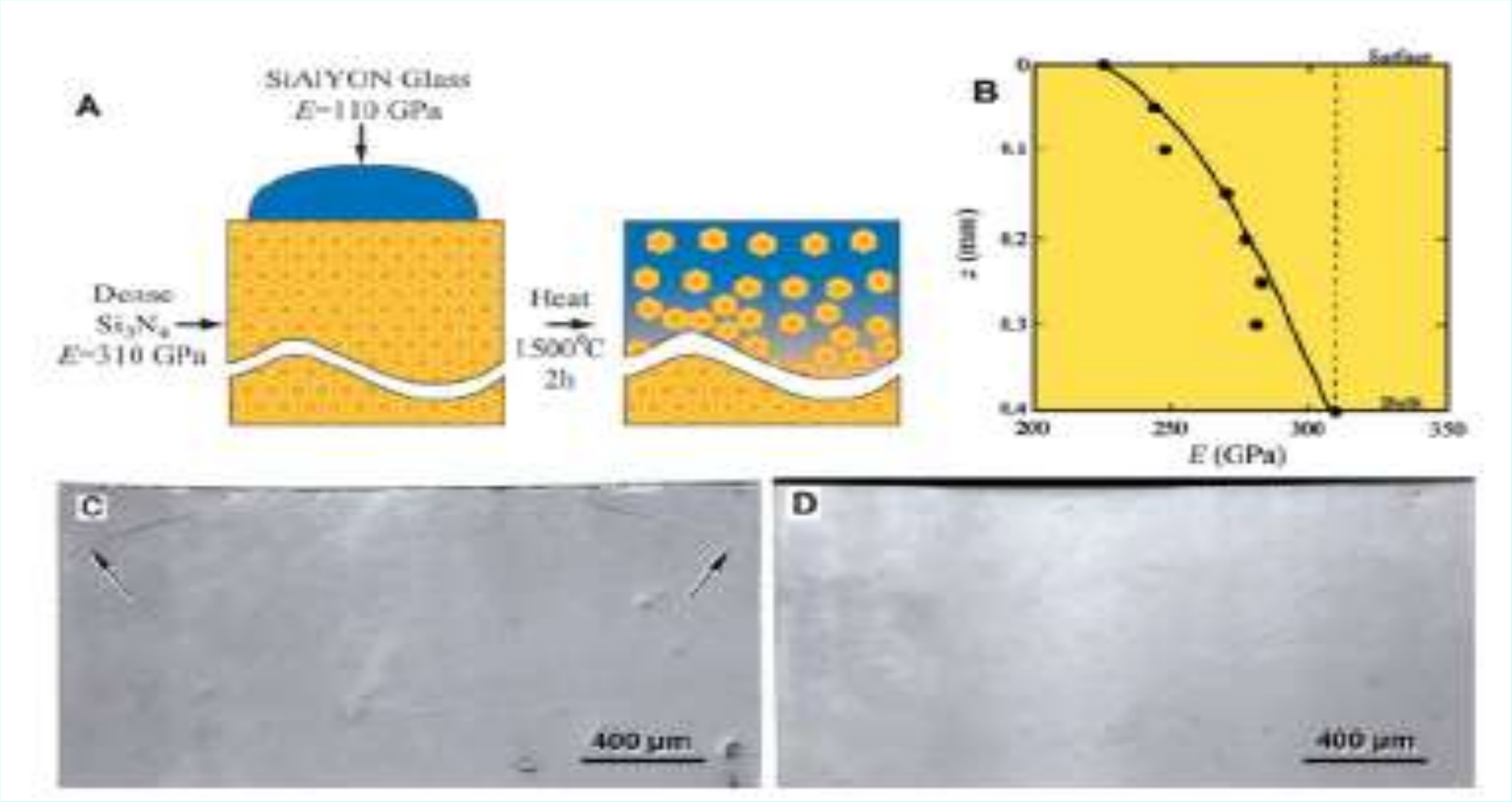
(1) Natural layered media



a) A typical pavement system

b) Piece-wise variation of layer moduli with depth

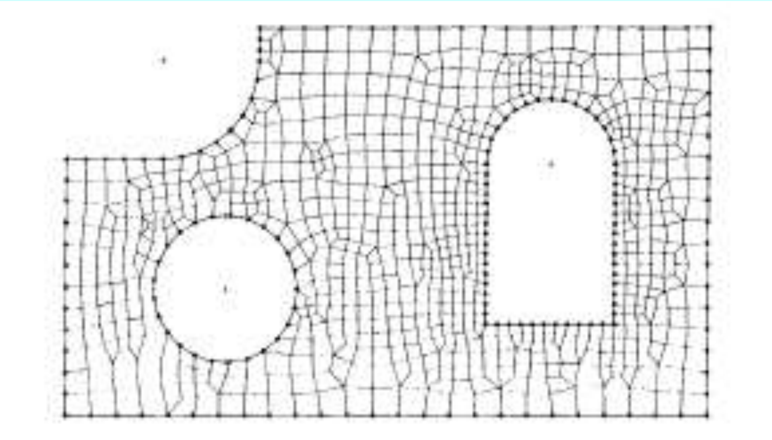
(2) The man-made materials, functionally graded materials



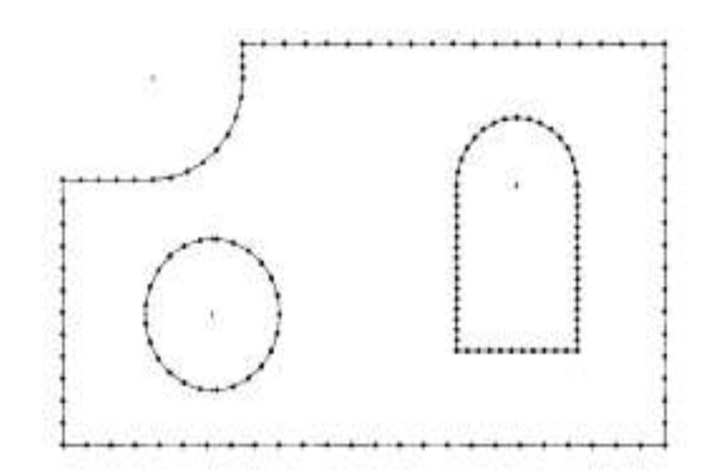
Suresh S. Graded materials for resistance to contact deformation and damage. *Science*,2001, 292:2447-51

The study of the fracture mechanics of layered elastic solids has always occupied a prominent position in solid mechanics.

The boundary element method (BEM) is now firmly established in many engineering disciplines and is increasingly seen as an effective numerical approach.



The discretized domain for FEM



The discretized boundary for BEM

The traditional BEM is based on Kelvin solution, which is the solution of homogeneous media of infinite extent subject to concentrated forces. So, many researchers developed the fundamental solutions of different materials.

2. Yue's solution of layered elastic solids of infinite extent

Journal of Elasticity **40:** 1–43, 1995. 1

On Generalized Kelvin Solutions in a Multilayered Elastic Medium*

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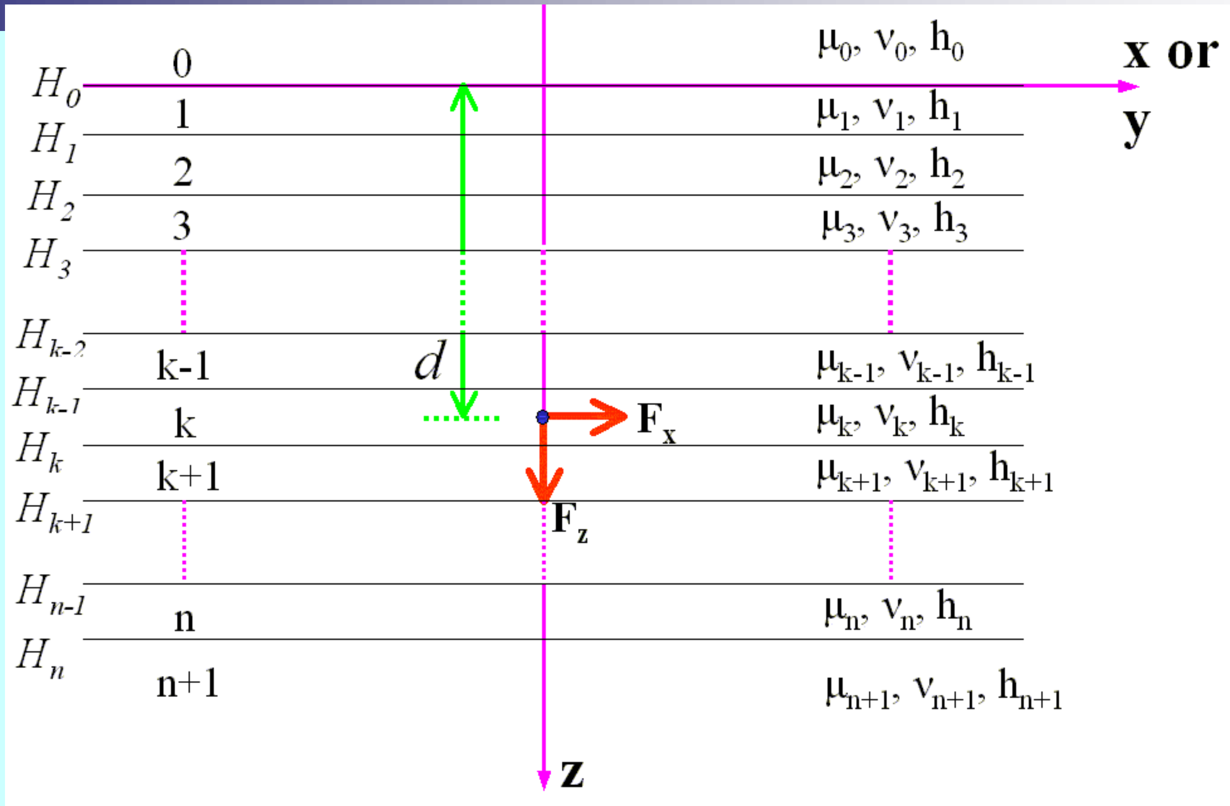
Abstract. This paper presents fundamental singular solutions for the generalized Kelvin problems of a multilayered elastic medium of infinite extent subjected to concentrated body force vectors. Classical integral transforms and a backward transfer matrix method are utilized in the analytical formulation of solutions in both Cartesian and cylindrical coordinates. The solution in the transform domain has no functions of exponential growth and is invariant with respect to the applied forces. The convergence of the solutions in the physical domain is rigorously and analytically verified. The solutions satisfy all required constraints including the basic equations and the interfacial conditions as well as the boundary conditions. In particular, singular terms of the generalized Kelvin solutions associated with the point and ring types of concentrated body force vectors are obtained in exact closed-forms via an asymptotic analysis. Numerical results presented in the paper illustrate that numerical evaluation of the solutions can be easily achieved with very high accuracy and efficiency and that the layering material inhomogeneity has a significant effect on the elastic field.



The layered solid in study can be briefly described as follows:

(1) The total number of the dissimilar layers is an arbitrary integer.

(2) The dissimilar homogeneous layers adhere to two upper and lower elastic solids.

(3) The interface between any two connected dissimilar layers is fully bonded.

The fundamental solution is characterized by the following:

(1)The classical Fourier integral transforms are employed to reduce the partial differential equations into algebraic equations.

(2) In the transform domain, the unknown coefficients of algebraic equations governing each layer are obtained using the interfacial and boundary conditions.

(3) The unknown coefficients can be analytically determined from boundary and interfacial conditions.

(4) The solution in the physical domain is obtained by integrating their transformed images and expressed in the form of inverse Hankel transform integrals.

The solution is also named as Yue's solution by some researchers:

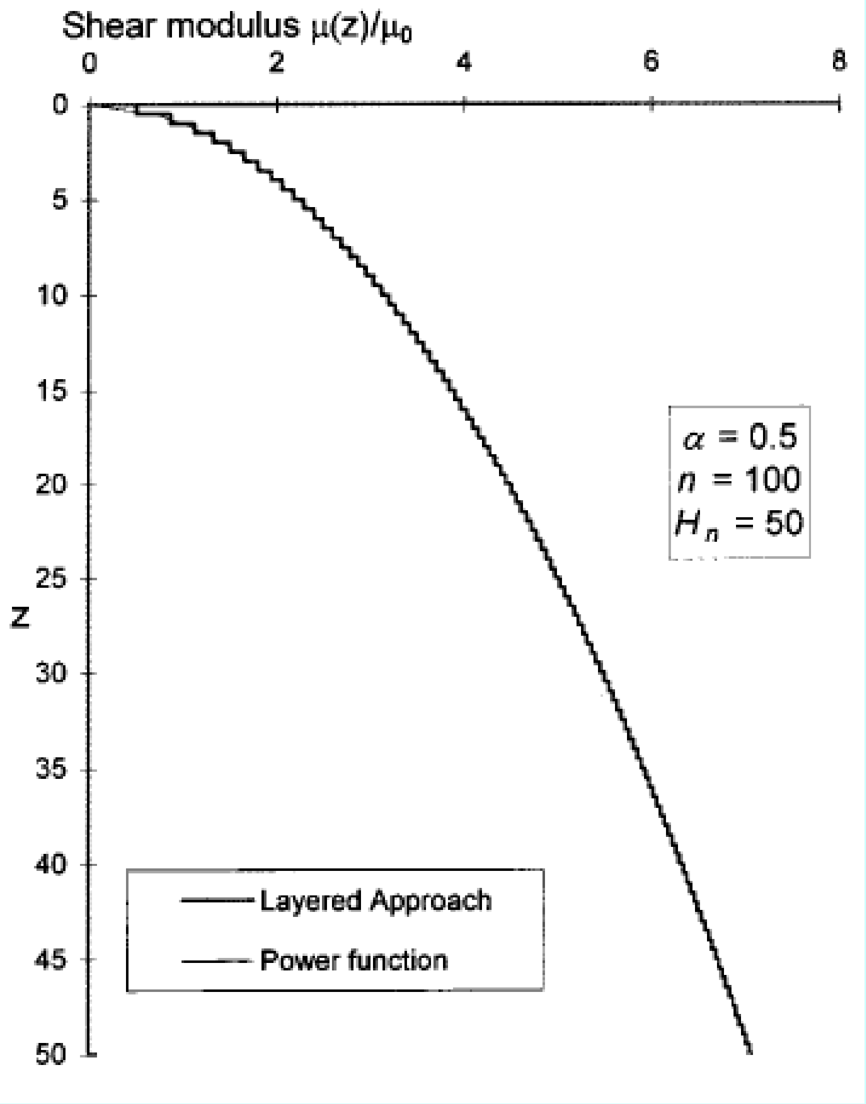
Maloney JM, Walton EB, Bruce CM, Van Vliet KJ. Influence of finite thickness and stiffness on cellular adhesion-induced deformation of compliant substrata. *Physical Review E*, 2008, 78(4):1-15.

Merkel R, Kirchgeßner N, Cesa C M, Hoffmann B. Cell force microscopy on elastic layers of finite thickness. Biophysical Journal, 2007, 93: 3314-3323.

Because the layer number is any arbitrary nonnegative integer, the graded media can be analyzed by using the layer discretization technique. The following figure shows the approximation.

The elastic parameter of a half-space varies in depth with any form.

The graded solid of semi-infinite extent is closely approximated by *n* bonded layers of elastic homogeneous media. The elastic parameters of each layer are presented according to the depth. A homogeneous half-space is bonded to the *n* layered solid. When the layer number *n* approach infinite, the solution of the graded solid can be obtained.



Yue's solution has been used to develop the BEM for the analysis of fracture mechanics in layered and graded solids.

Xiao HT and Yue ZQ. A generalized Kelvin solution based BEM for contact problems of elastic indenter on functionally graded materials. *CMES: Computer Modeling in Engineering & Sciences*, 2009, 52(2):159-179.

Yue ZQ, Xiao HT and Pan E. Stress intensity factors of square crack inclined to interface of transversely isotropic bi-material. *Engineering Analysis with Boundary Elements*. 2007,31:50-56.

Xiao HT, Yue ZQ, Tham GL. Stress intensity factors for penny-shaped cracks perpendicular to graded interfacial zone of bonded bi-materials. *Engineering Fracture Mechanics*, 2005, 72:121-143.

Xiao HT, Yue ZQ, Tham GL. Analysis of an elliptical crack parallel to graded interfacial zone of bonded bi-materials. *Mechanics of Materials*, 2005, 37:785-799.

Xiao HT, Yue ZQ, Tham GL, Lee CF. Analysis of elliptical cracks perpendicular to the interface of two joined transversely isotropic solids. *International Journal of Fracture*. 2005, 133:329-354.

Yue ZQ, Xiao HT, Tham GL, Lee CF and Yin JH. Stresses and displacements of a transversely isotropic elastic halfspace due to rectangular loadings. *Engineering Analysis with Boundary Elements*. 2005, 29: 647-671.

Yue ZQ, Xiao HT, Tham GL, Lee CF, Pan E. Boundary element analysis of three-dimensional crack problems in two joined transversely isotropic solids. *Computational Mechanics*. 2005, 36:459-474.

Yue ZQ, Xiao HT and Tham GL. Elliptical crack normal to functionally graded interface of bonded solids. *Theoretical and Applied Fracture Mechanics*, 2004, 42:227-248.

Yue ZQ, Xiao HT and Tham LG. Boundary element analysis of crack problems in functionally graded materials. *International Journal of Solids and Structures*. 2003, 40(13-14): 3273-3291.

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Xiao HT, Yue ZQ and Zhao XM. A generalized Kelvin solution based method for analyzing elastic fields in heterogeneous rocks due to reservoir water impoundment. *Computers & Geosciences*, 2012, 43:126-136.

Xiao HT and Yue ZQ. Elastic fields in two joined transversely isotropic media of infinite extent as a result of rectangular loading. *Int. J. Numer. Analy. Meth. Geomech.*, 2013,37:247-277.

Xiao HT and Yue ZQ. A three-dimensional displacement discontinuity method for crack problems in layered rocks. *Int. J. Rock Mech. Min. Sci.*, 2011, 48:412-420.

Xiao HT and Yue ZQ. Stress and displacement in FGMs of semi-infiite extent induced by rectangular loading. *Materials*, 2012,5(2):210-226.

Xiao HT, Xie YY, Yue ZQ. Analysis of square-shaped crack in layered halfspace subject to uniform loading over rectangular surface area. *CMES: Computer modeling in Engineering & Sciences*. 2015, 109-110(1):55-80.

Xiao HT and Yue ZQ. *Fracture mechanics in layered and graded solids: analysis using boundary element methods*. Berlin: De Gruyter & Higher Education Press, 2014.

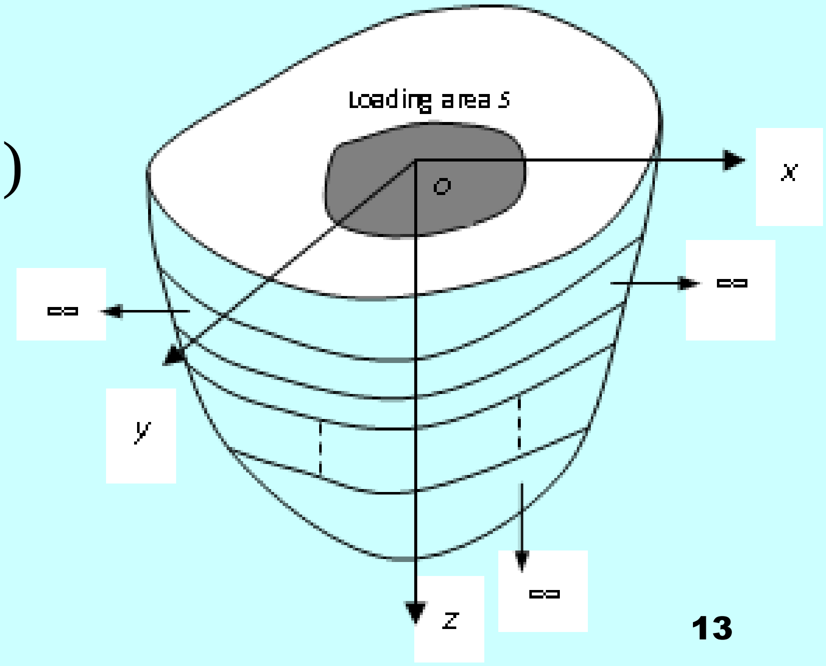
3. Numerical methods for analyzing the crack problem in a layered half-space

3.1 *LayerSmart3D:Numerical method for elastic fields in a layered medium*

We have developed a numerical method for calculating elastic fields of a heterogeneous medium with depth using Yue's solution. The stresses $\sigma_{ij}(Q)$ and displacements $u_i(Q)$ at any point Q of the layered medium are described as

$$\sigma_{ij}(Q) = \int_S \sigma^*_{ijk}(Q,P)t_k(P)d\Gamma(P) \quad (1)$$

$$u_i(Q) = \int_S u^*_{ik}(Q,P)t_k(P)dS(P) \quad (2)$$



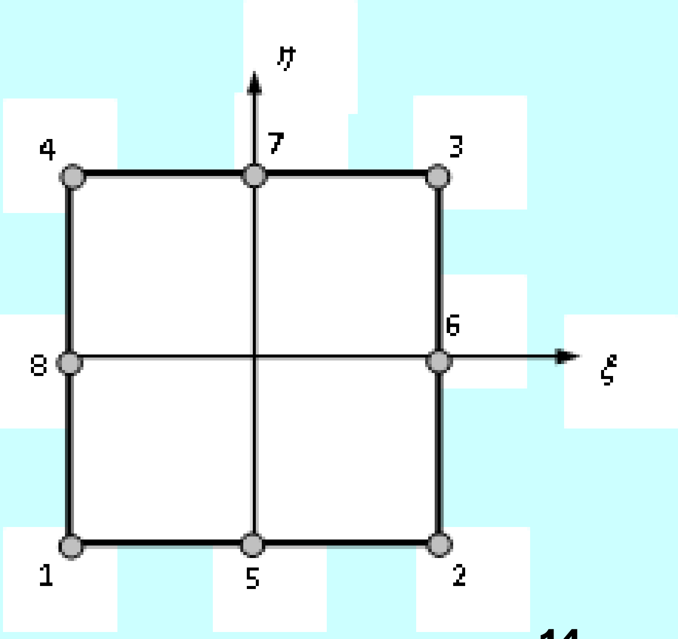
where $\sigma^*_{ijk}(Q,P)$ are stresses of Yue's solution for the field point Q due to the unit force along the k direction at the source point P, $t_k(P)$ is the traction at the source point P, and the integral domain is the loading area.

The quadrilateral element is employed to discretize the boundary surfaces. The local coordinate system is attached at every element.

The loading domain S is discretized into ne elements. Expressions (1) and (2) are written in the forms

$$\sigma_{ij}(Q) = \sum_{e=1}^{ne} \int_{S_e} \sigma^*_{ijk}(Q,P)\, t_k(P)\, dS(P) \quad (2)$$

$$u_i(Q) = \sum_{e=1}^{ne} \int_{S_e} u^*_{ik}(Q,P)\, t_k(P)\, dS(P) \quad (3)$$



In global and local coordinate systems, there are the following transform relationships of coordinate and traction values:

$$x_i = \sum_{l=1}^{4\text{ to }8} N_l(\xi,\eta)x_i^l \quad t_i = \sum_{l=1}^{4\text{ to }8} N_l(\xi,\eta)t_i^l$$

So, we have the following discretized forms

$$\sigma_{ij} = \sum_{e=1}^{ne}\sum_{l=1}^{4\text{ to }8} t_k^l \int_{-1}^{1}\int_{-1}^{1} \sigma_{ijk}^{*} N_l \det \mathbf{J} d\xi d\eta \tag{4}$$

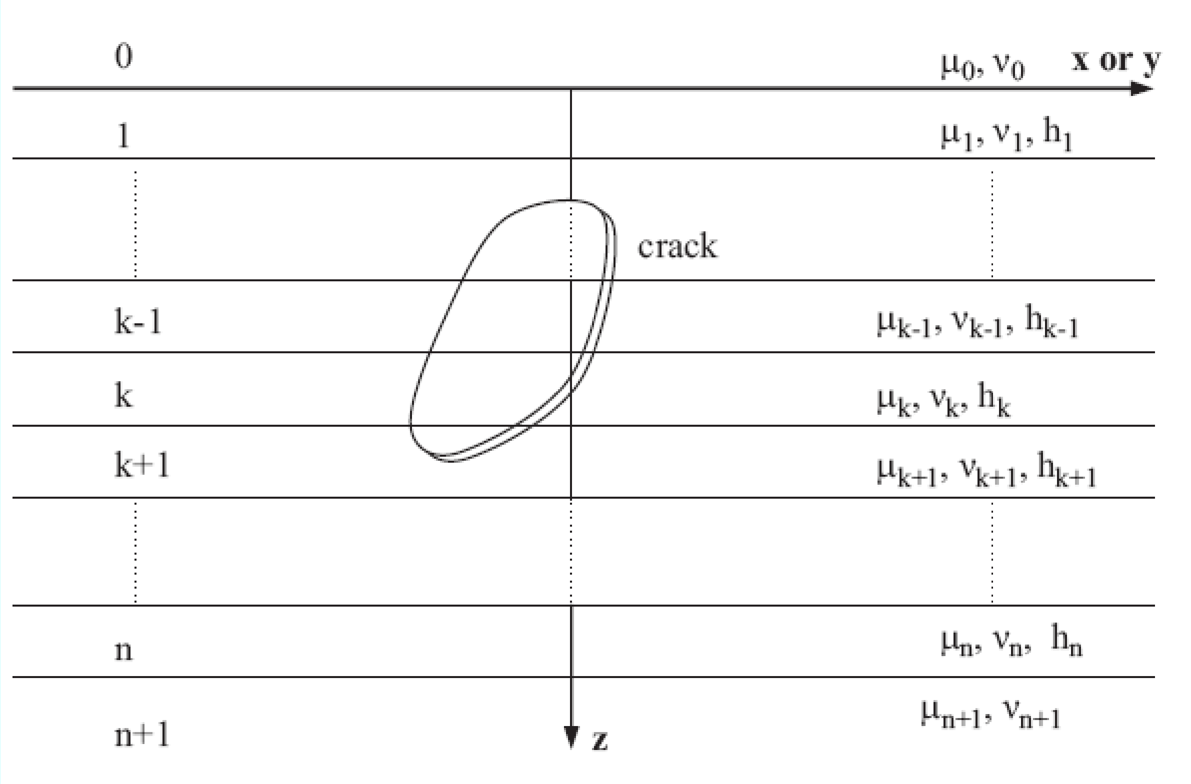
$$u_i = \sum_{e=1}^{ne}\sum_{l=1}^{4\text{ to }8} t_k^l \int_{-1}^{1}\int_{-1}^{1} u_{ik}^{*} N_l \det \mathbf{J} d\xi d\eta \tag{5}$$

where $\boldsymbol{J}$ is the matrix of the Jacobian transformation and det $\boldsymbol{J}$ is the corresponding determinant. By using the coordinate transform, the integral domains in expressions (4) and (5) are changed into the domain in the local coordinates.

The integrals of expressions (4) and (5) are executed in the local coordinates and are calculated by using the regular Gaussian quadrature.

3.2 *LayerDDM3D: DDM for analyzing crack problems in a layered medium*

We have developed a new displacement discontinuity method (DDM) for the analysis of crack problems in layered strata of infinite extent. This approach is also based on Yue's solution and the corresponding computer code was written in FORTRAN.



Cracks in the multilayered solids of infinite extent

In global coordinates, the discontinuous displacements of the crack surfaces are defined as .

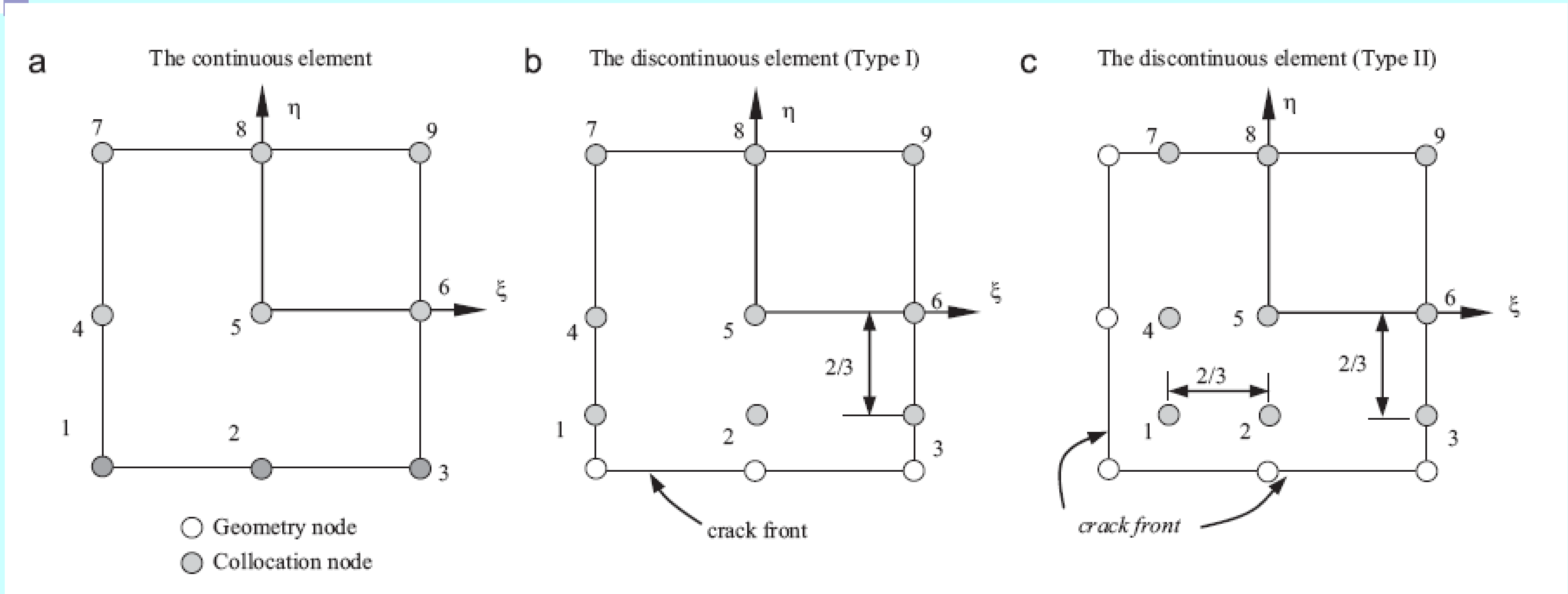
$$\Delta u_j\left(Q_{\Gamma_C^+}\right)=u_j\left(Q_{\Gamma_C^+}\right)-u_j\left(Q_{\Gamma_C^-}\right) \qquad (6)$$

u_j is a displacement along the j direction. The traction on the crack surface is defined as $t_j\left(P_{\Gamma_C^+}\right)$. The basic equations of Yue's solution based DDM are as follows

$$t_j\left(P_{\Gamma_C^+}\right)+n_i\left(P_{\Gamma_C^+}\right)\int_{\Gamma_C^+} T_{ijk}^*\left(P_{\Gamma_C^+},Q\right)\Delta u_k\left(Q\right)d\Gamma\left(Q\right)=0 \qquad (7)$$

where $n_i\left(P_{\Gamma_C^+}\right)$ is the cosine of normal direction at the source point, $T_{ijk}^*\left(P_{\Gamma_C^+},Q\right)$ is the kernel function, which can be calculated by using the traction of Yue's solution.

The 9-noded elements are used to discretize the crack surface. There are two types of elements including continuous and discontinuous elements.

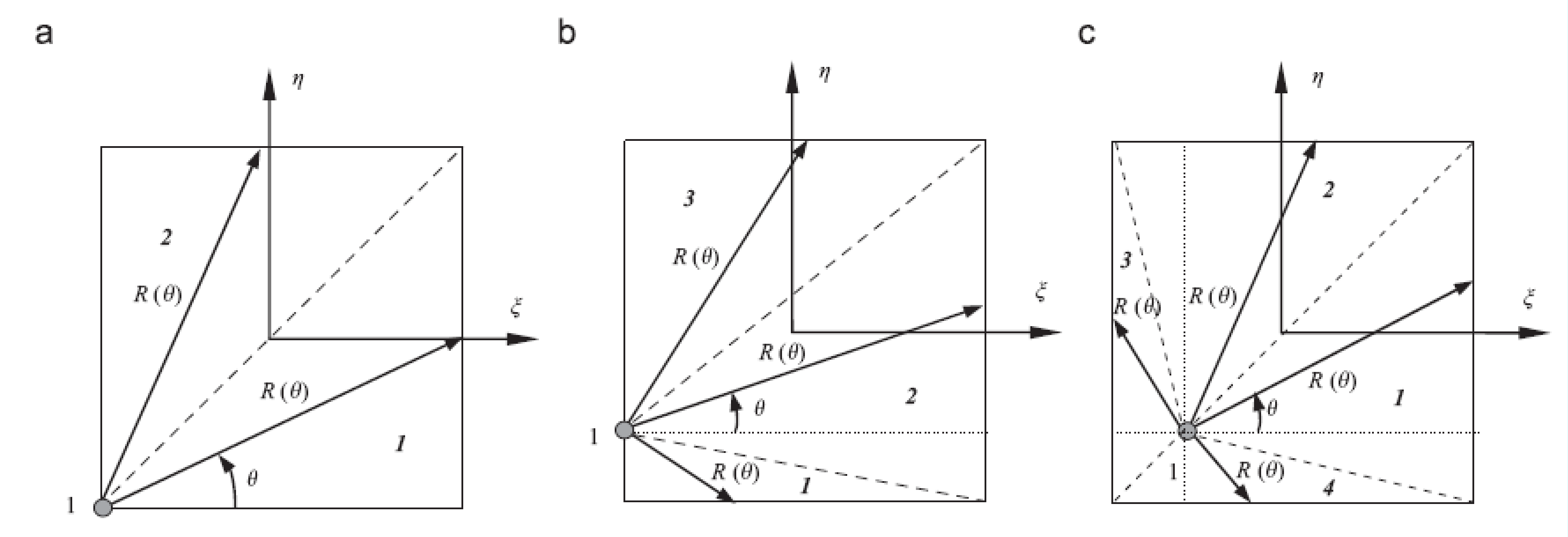


Eq. (7) can be discretized into the following form:

$$t_j(P_{\Gamma^+}) + n_i(P_{\Gamma^+}) \sum_{e=1}^{NE} \int_{\Gamma_e^+} T^*_{ijk}(P_{\Gamma^+}, Q)\Delta u_k(Q)\, d\Gamma(Q) = 0, \quad i,j,k = x,y,z, \qquad (8)$$

Eq. (8) contains regular and hyper-singular integrals. The regular integrals can be calculated using Gaussian quadrature. The hyper-singular integrals can be calculated using Kutt's quadrature.

In calculating the hypersingular integral, the element should be divided into several triangle domains as follows:

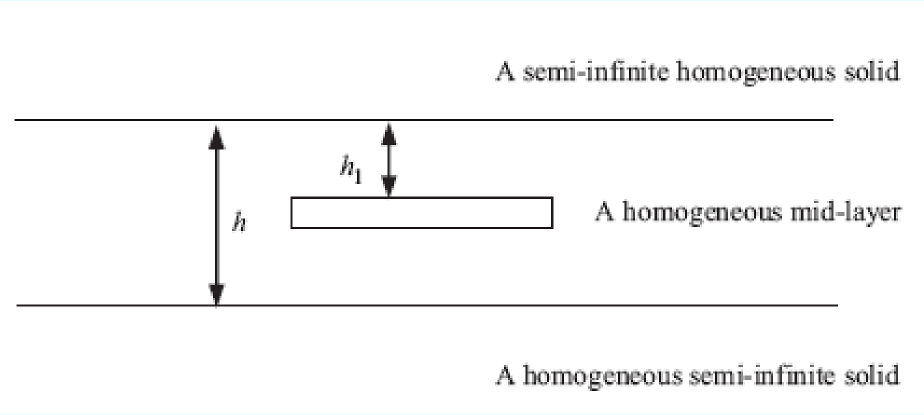


The hypersingular integral in Eq. (8) can be further written as

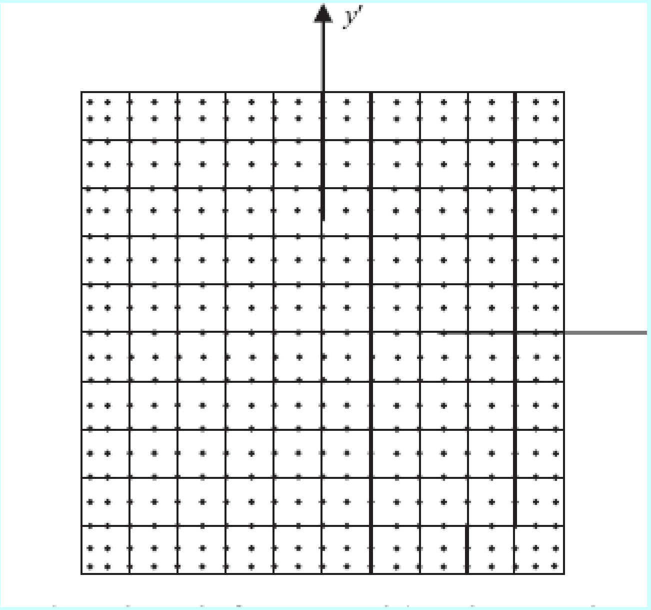
$$\sum_{m} \int_{\theta_1}^{\theta_2} \int_{0}^{R(\theta)} T^{*}_{ijk}\left[P_{\Gamma^+}(\xi^c,\eta^c),Q(r,\theta)\right] g(r,\theta)\phi_l(r,\theta)J(r,\theta) r \, dr \, d\theta,$$

In the numerical examples, Kutt's 20-point quadrature is used in the finite-part integral with respect to *r*, and 20 Gaussian points for the regular outer integral with respect to θ.

When the cracked layered solids of infinite extent are subject to the loading only on the crack surface, only the crack surface can be discretized.



A squared crack in the layered solids of infinite extent



The mesh of a square crack

3.3 The superposition principle of fracture mechanics

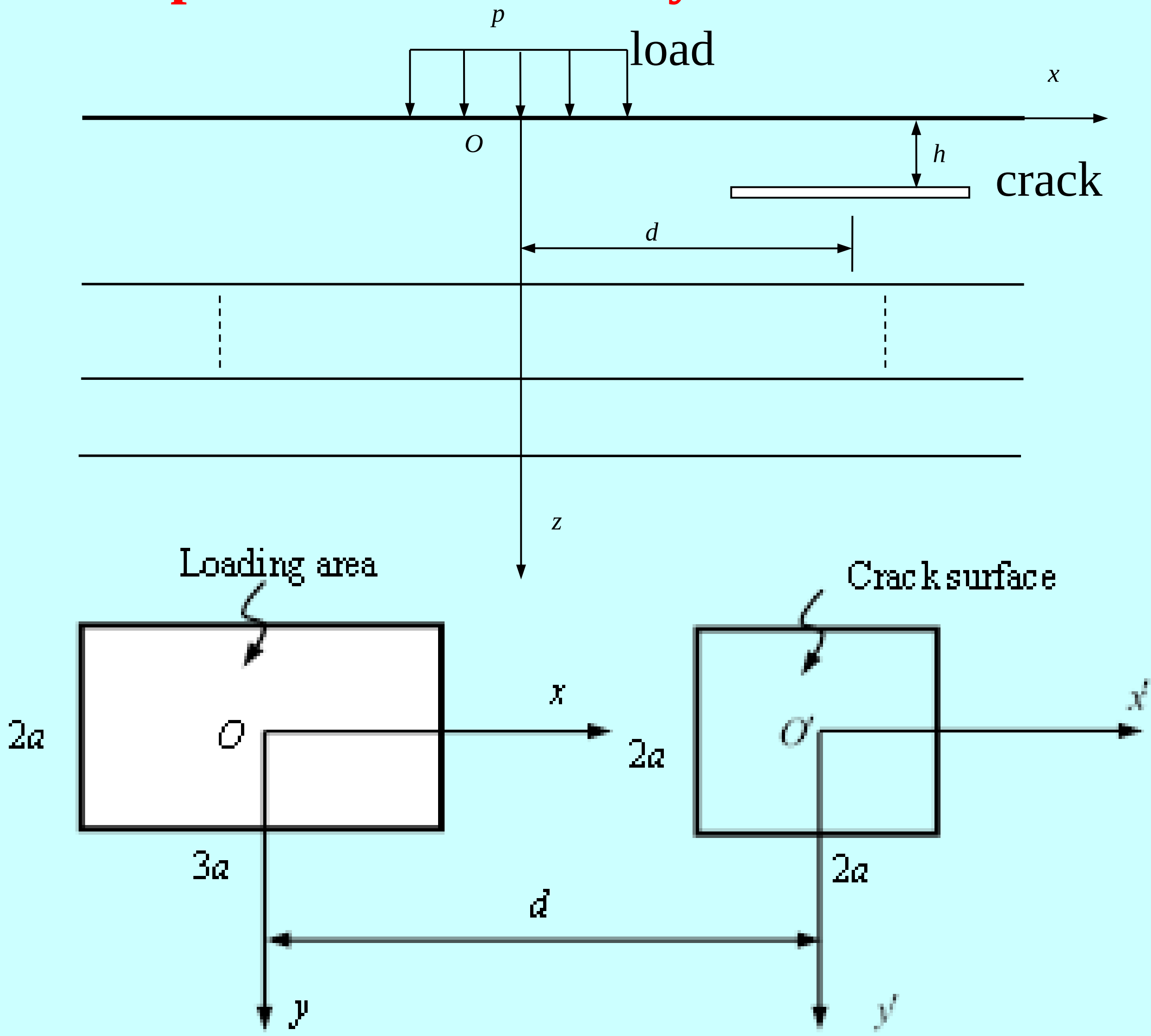
The above-mentioned two numerical methods and the superposition principle in fracture mechanics are utilized to analyze crack problems in a layered medium. The analytical process is described as follows:

(1) LayerSmart3D is employed to obtain the stress fields of a layered medium without a crack under the action of rectangular loadings on the boundary surface.

(2) Using the superposition principle, the tractions, which are equal to the stress calculated above and have opposite directions, are then loaded on the crack surfaces in the layered medium without rectangular loadings on the boundary surface.

(3) LayerDDM3D is employed to obtain the discontinuous displacements of the crack surfaces under the action of the above tractions.

3.4 The crack problems to be analyzed



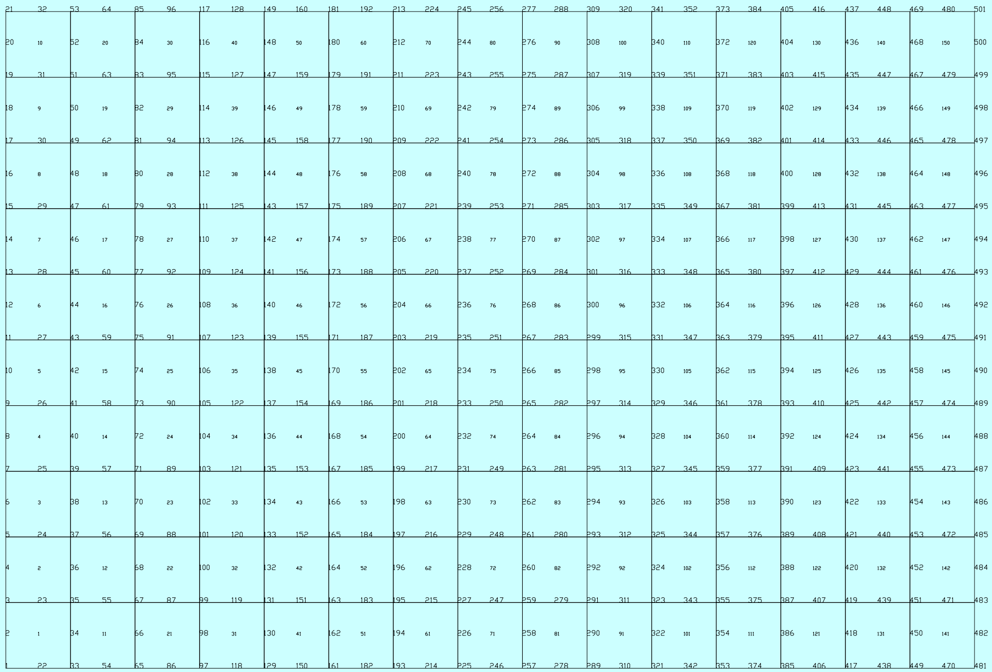
Crack in a layered half-space subject to loadings on the boundary surface

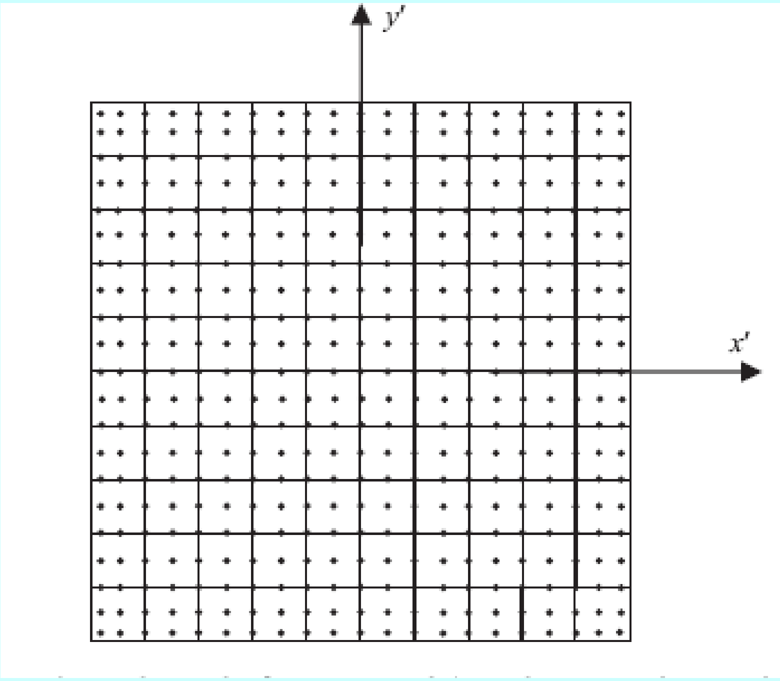
In an infinite layered medium, the elastic modulus of the upper semi-infinite medium is given an extremely small value and the Poisson's ratio of the medium, i.e.,

$$E_0 = 10^{-15}\,\text{MPa},\ \ v_0 = 0.3$$

In this way, the fundamental solution of a layered medium of semi-infinite extent is obtained.

The discretized loading area and crack surface



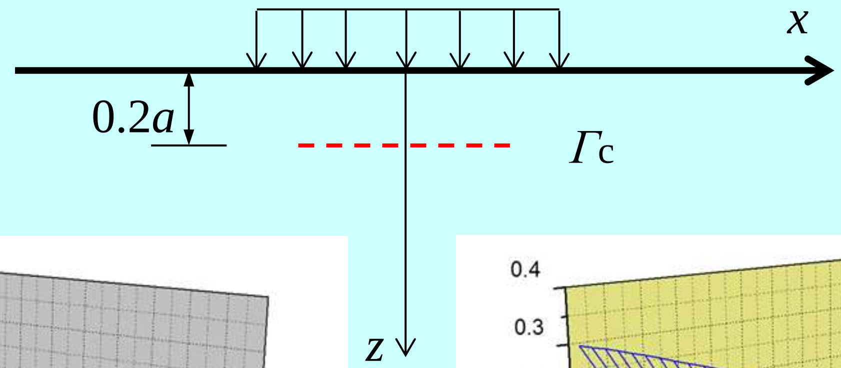


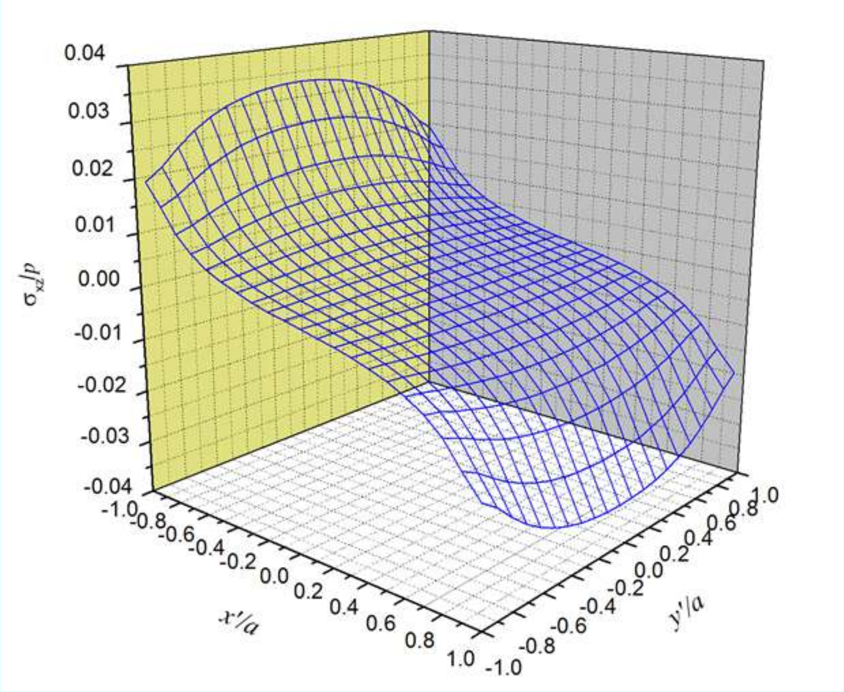
The discretized loading area:
501 nodes and 150 elements

The discretized crack surface:
441 nodes and 100 elements

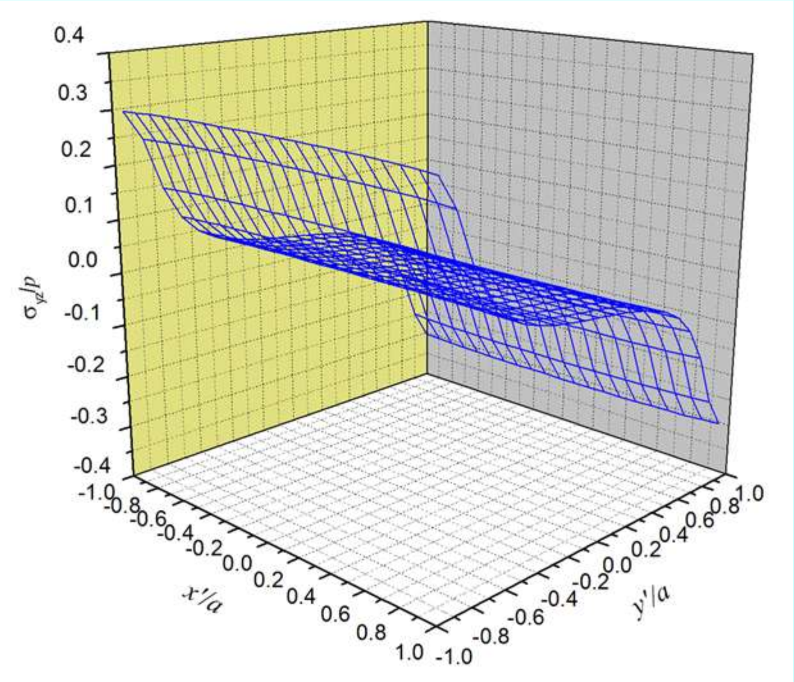
4. Stress fields of a homogeneous half-space without cracks

4.1 The non-crack area located directly below the loading area



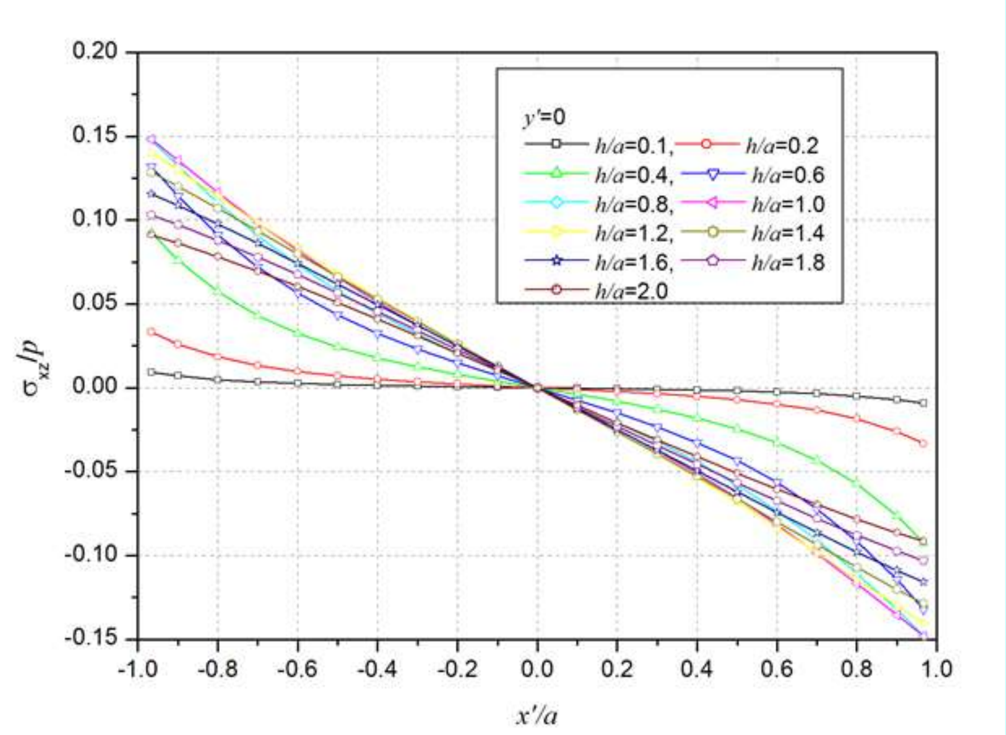


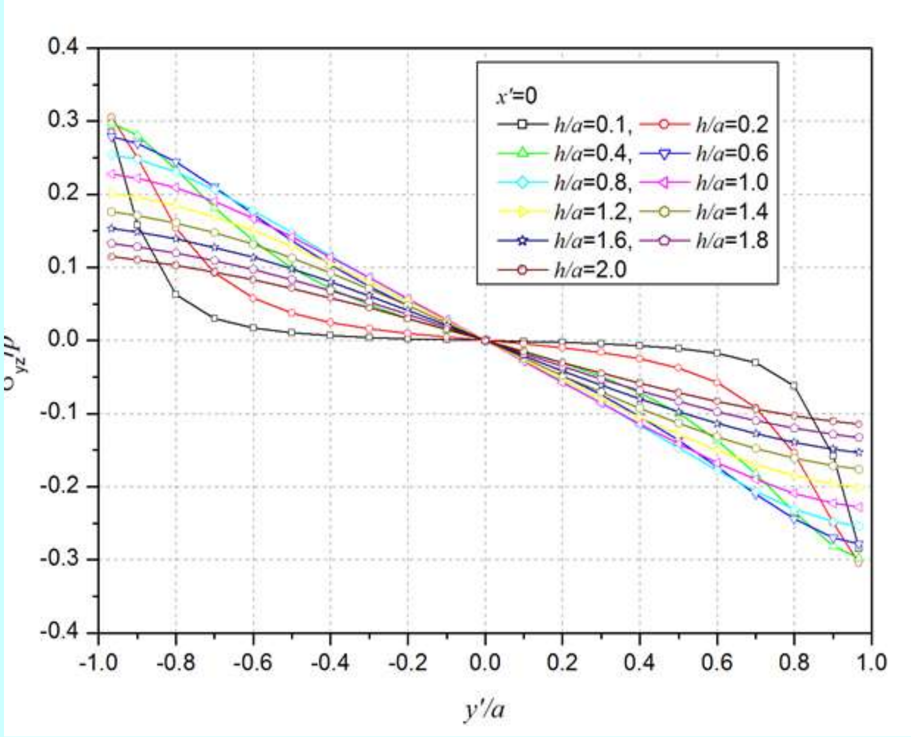
Stress σ_{xz} on Γ_c at d=0 and h=0.2a



Stress σ_{yz} on Γ_c at d=0 and h=0.2a

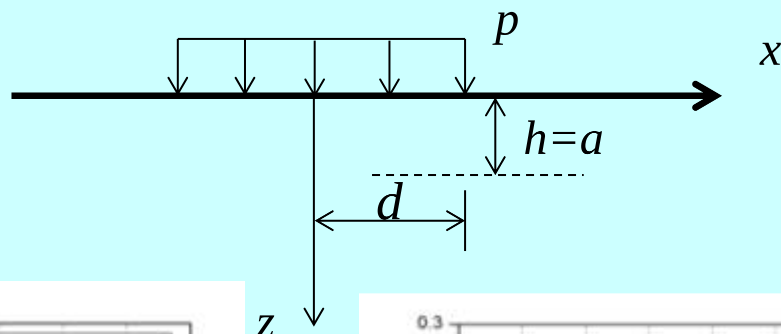
The stress variations with the depth *h*.

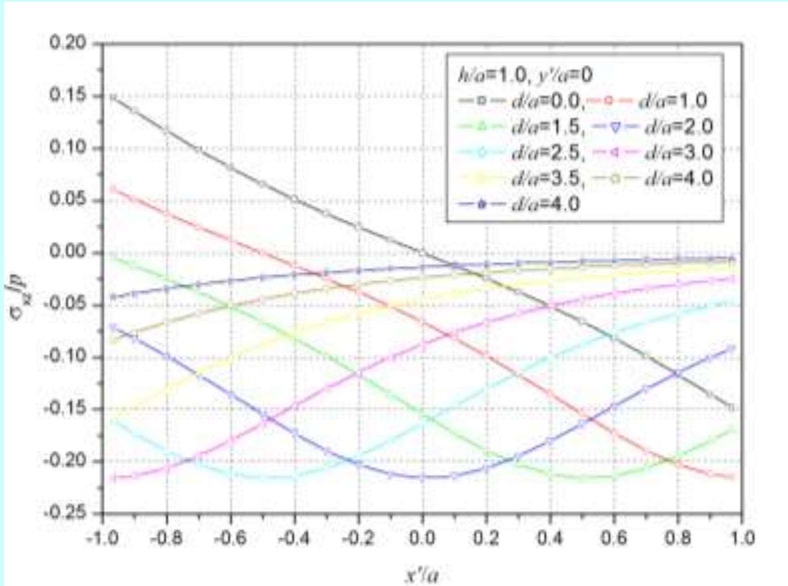


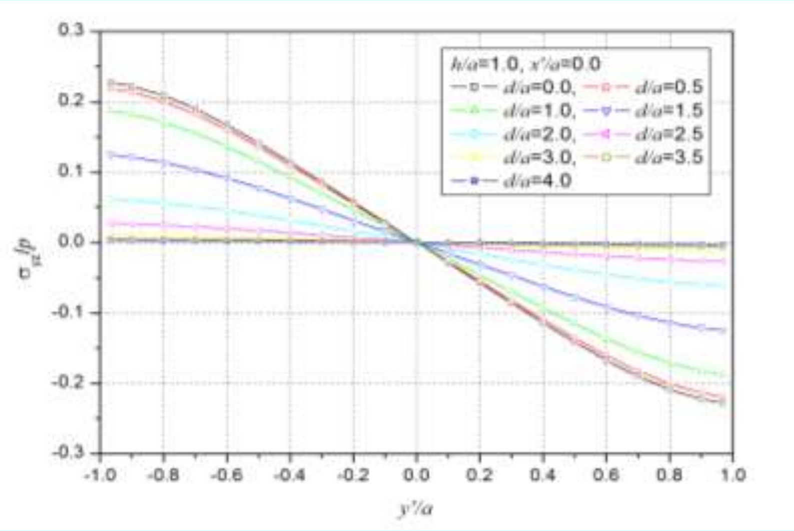


It can be found that the absolute values of the stresses decrease with depth *h* increasing.

4.2 ***The non-crack area located at different positions along the x axis***







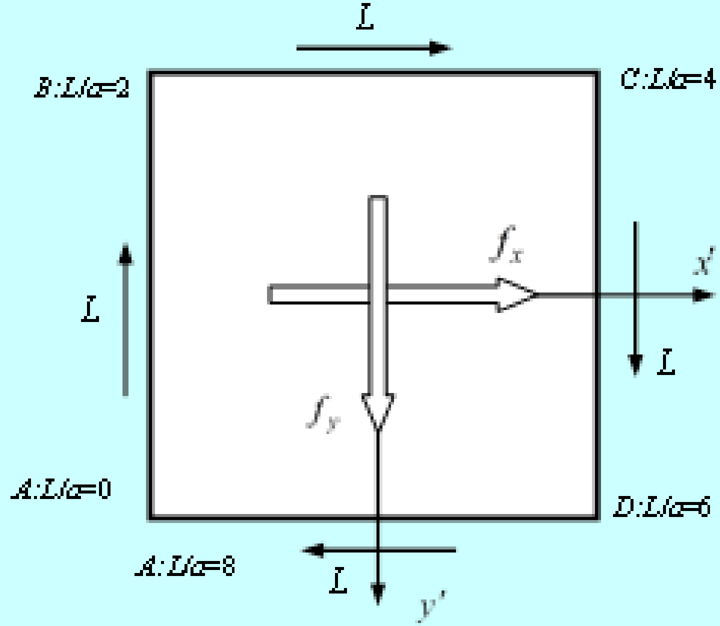
Stress σ_{xz} on Γc at $h=a$

Stress σ_{yz} on Γc at $h=a$

5. Analytical methods of square-shaped cracks in layered media

5.1 Calculating the discontinuous displacements of crack surfaces

Using the superposition principle, the absolute values of the tractions on the crack surfaces in a layered half space without the surface force *p* are equal to the ones of the shear stresses on the area in a layered half-space without a crack whilst the directions of the tractions and the shear stresses are completely opposite.



The crack surfaces are subjected to the following loadings:

$$f_x = -\sigma_{xz} \quad \text{and} \quad f_y = -\sigma_{yz}$$

The stresses are calculated using the code LayerSmart3D. Thus, discontinuous displacements are calculated using the code LayerDDM3D.

5.2 Calculating stress intensity factors and analyzing crack growth

The stress intensity factor can be calculated using the discontinuous displacements of the crack surface and the following formula:

$$K_{\mathrm{I}} = \frac{E}{4(1-\nu^2)}\sqrt{\frac{\pi}{2r}}\Delta u_{z'}(r,\theta = \pm\pi, \varphi = -\pi/2),$$

$$K_{\mathrm{II}} = \frac{E}{4(1-\nu^2)}\sqrt{\frac{\pi}{2r}}\Delta u_{y'}(r,\theta = \pm\pi, \varphi = -\pi/2),$$

$$K_{\mathrm{III}} = \frac{E}{4(1+\nu)}\sqrt{\frac{\pi}{2r}}\Delta u_{x'}(r,\theta = \pm\pi, \varphi = -\pi/2),$$

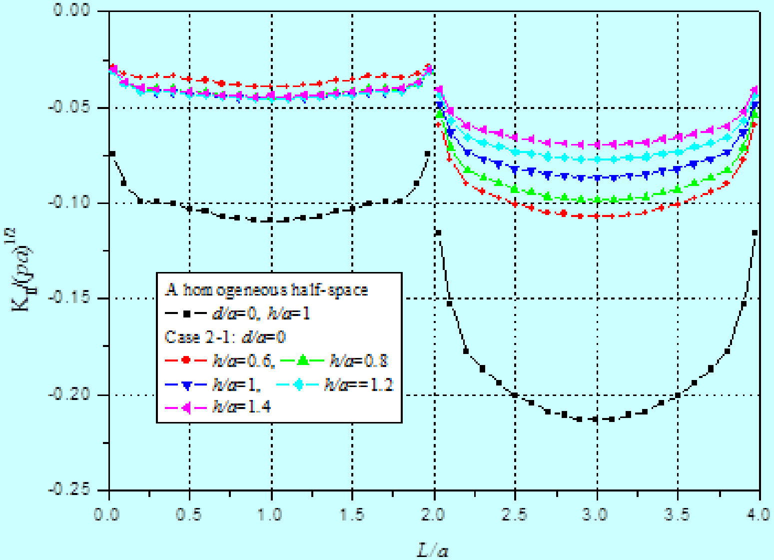
Growth of a crack in elastic solids subject to complex stress states can be assessed using the minimum strain energy density criterion. The criterion states that local instability is assumed to occur when the local minimum energy factor S_{min} reaches a critical value S_{cr}.

The strain energy density can be calculated using the formulae:

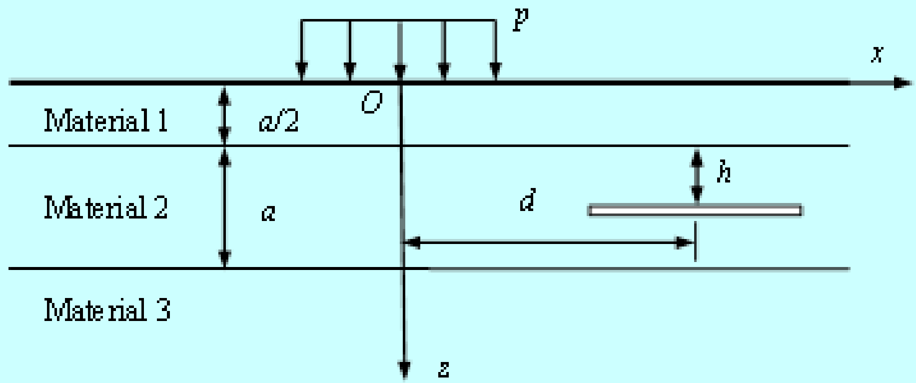
$$S(\theta) = a_{22}(\theta)K_{\mathrm{II}}^2 + a_{33}(\theta)K_{\mathrm{III}}^2$$

6. Numerical examples

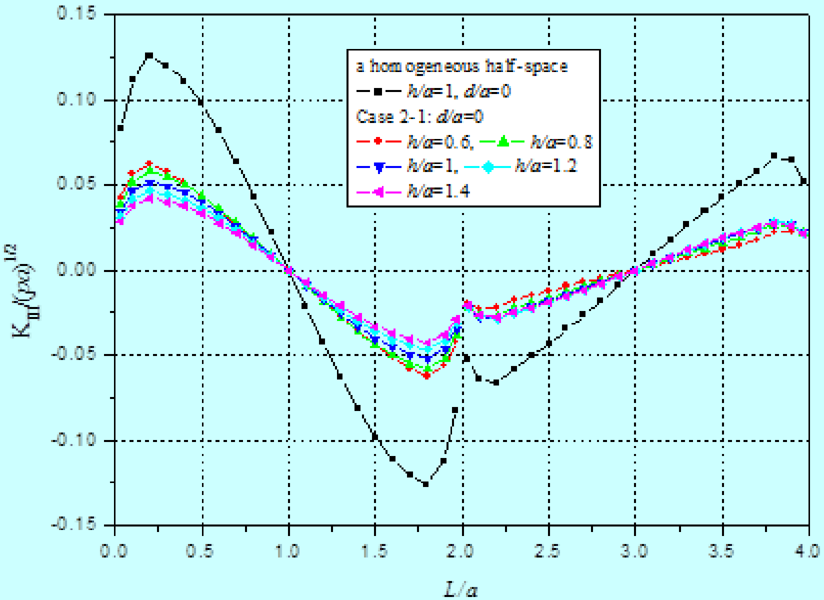
K_{II} Variations with h



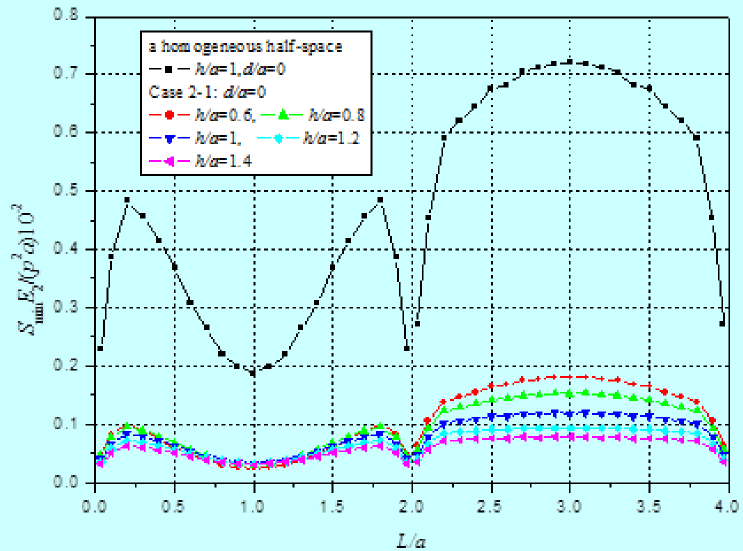
Case 1: $E_1 = E_3 = 2E_2$



K_{III} Variations with h

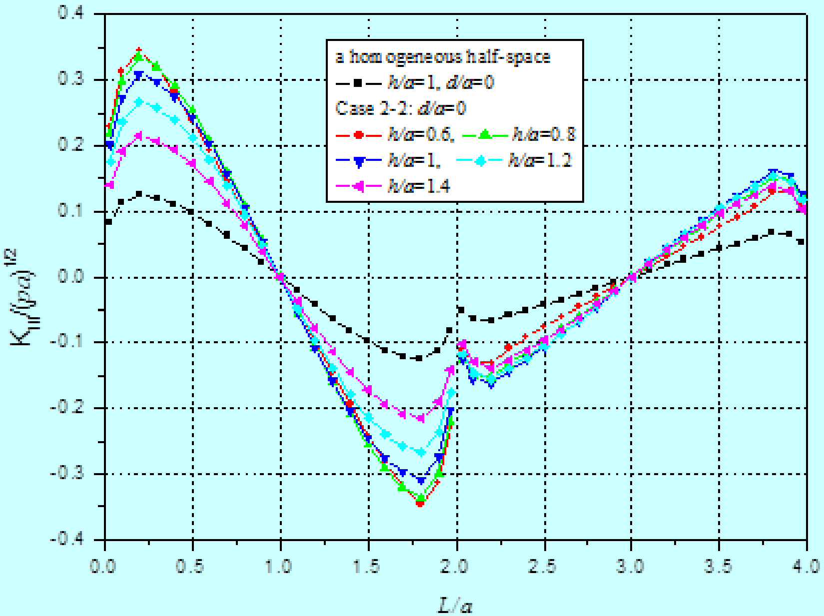


S_{min} Variations with h

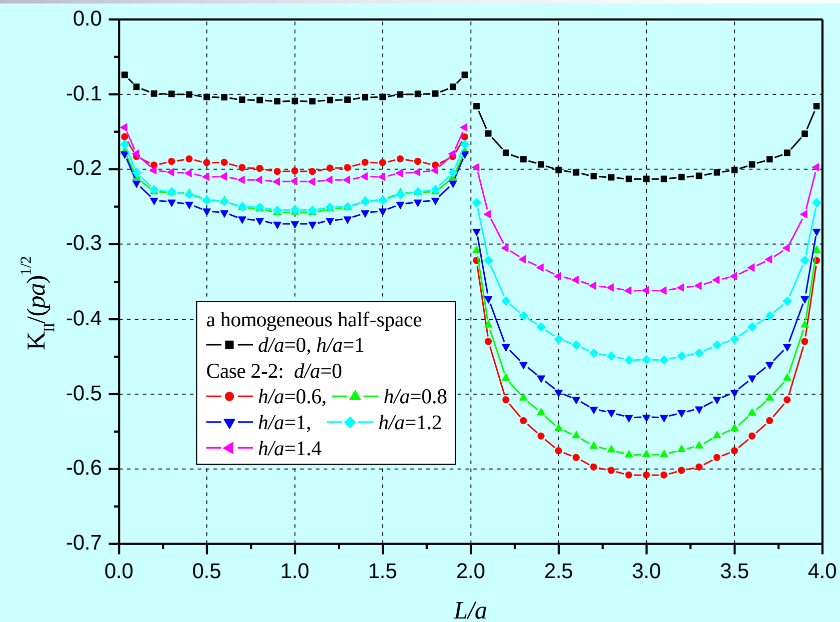


Case 2: $E_1 = E_3 = E_2 / 2$

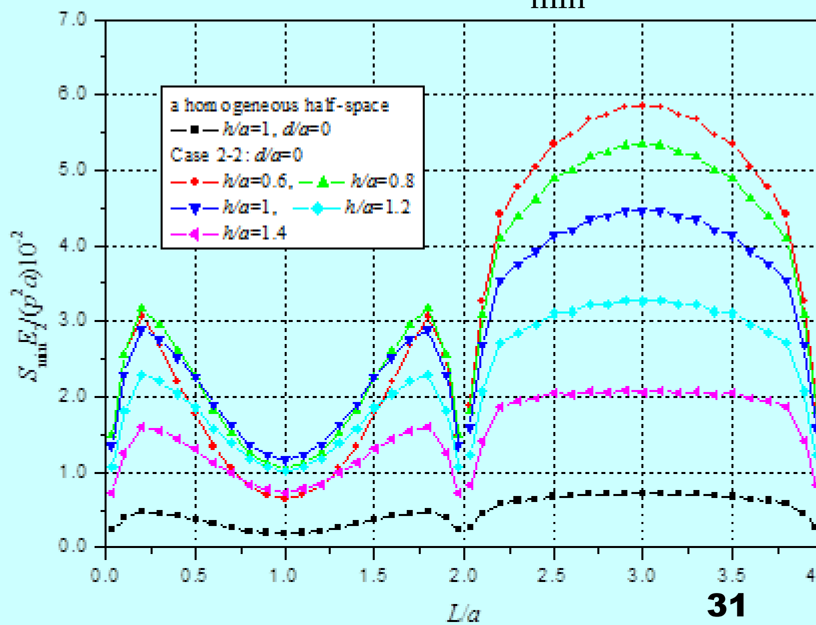
Variations of K_{III} with h



Variations of K_{II} with h



Variations of S_{min} with h



7. Recommendations

The two proposed numerical method, together with the superposition principle of fracture mechanics, can be applied to the analysis of cracks in a half-space subject to complex loadings.

Of course, the proposed methods can be used for the analysis of crack problems in a layered halfspace with arbitrary elastic moduli with depth.

Thank you!