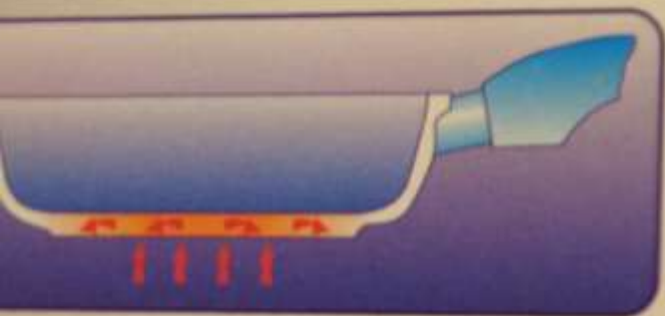

RoseBakes.com

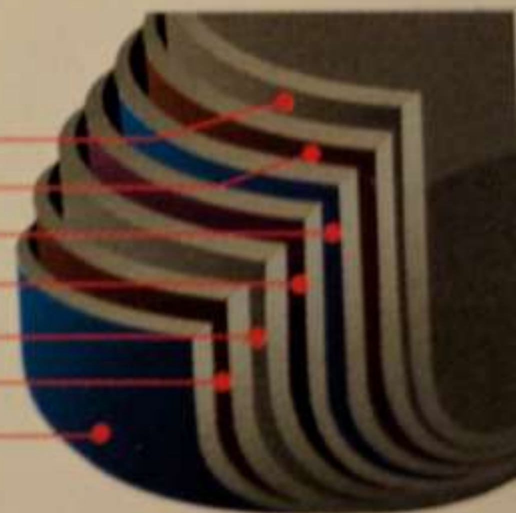
king surface before initial use; it will make the coating last longer.
g pan on heat for long periods of time, this will prevent staining and ensures a

coating, do not leave salty foods on cooking surface for extended periods of t
as or sharp kitchen tools
pan after using and store in a dry place



HOT SPOT-FREE BASE

VITH HEAVY-DUTY CASTING BODY



Item No.800064

Non-stick Pan

An Overview of Pavement Forward and Inverse Analyses

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The University of Hong Kong
Thursday 12/15/16

Outline

--- Introduction of MultiSmart3D

--- Other Methods

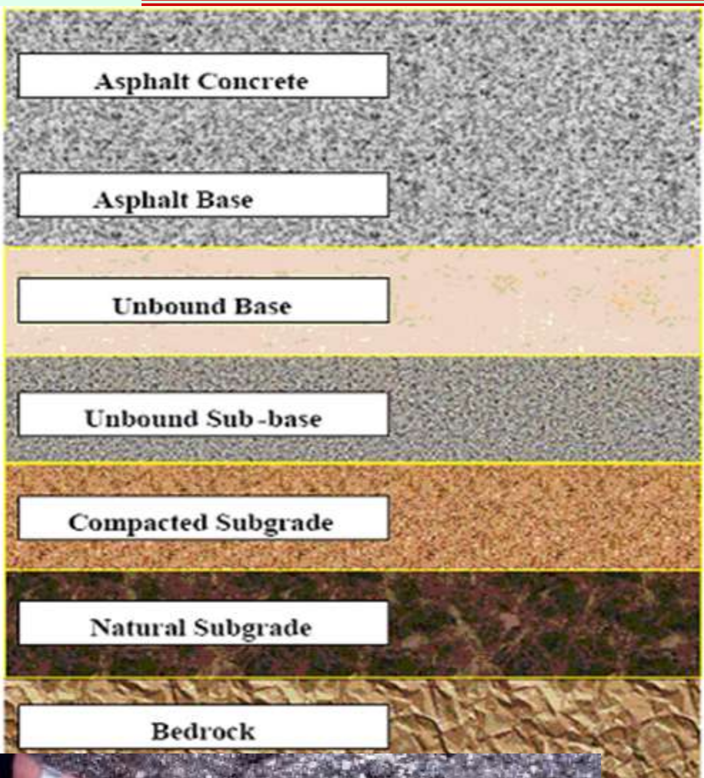
--- Effect of Transverse Isotropy

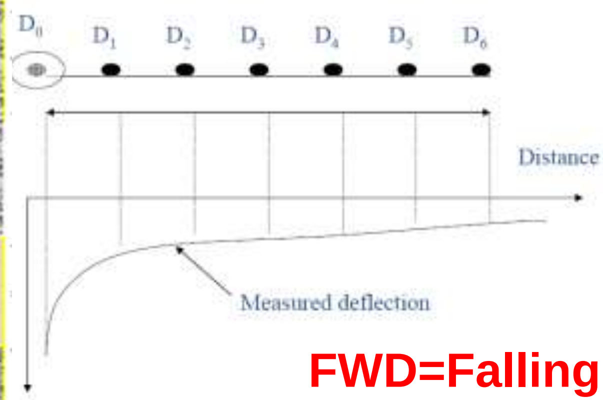
--- Moduli Backcalculation

--- Forward/Inverse with Bonding

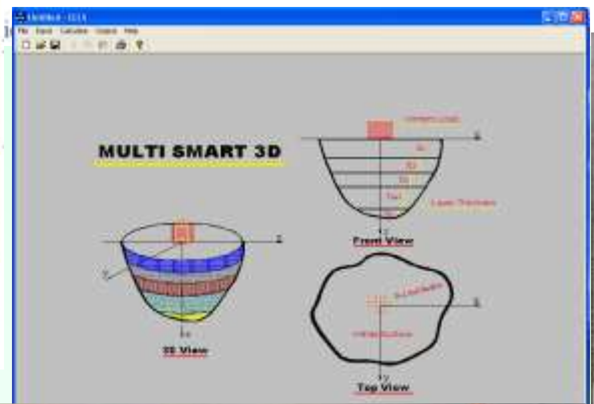
--- Conclusions

Why Multilayered Pavement Analyses?





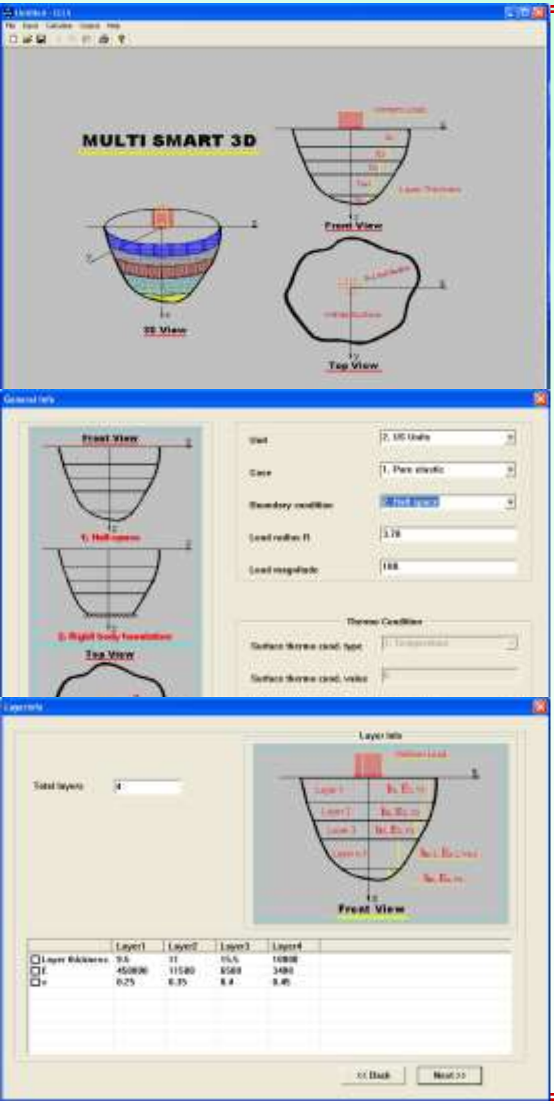
FWD=Falling weight deflectometer

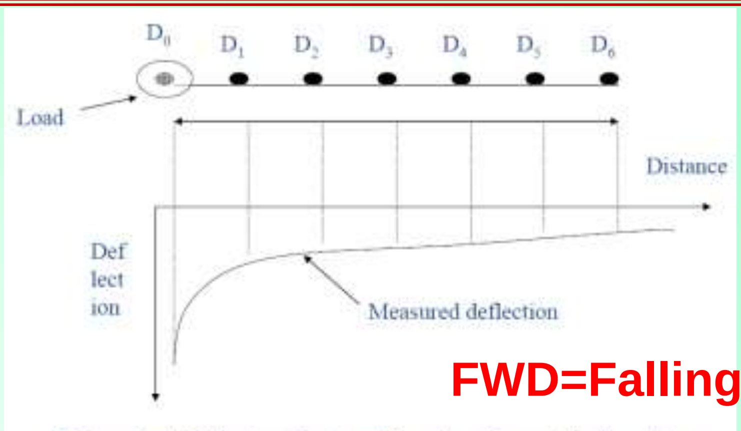


Fatigue, low temp, perm

From MultiSmart3D to BackGenetic3D???

CHEVRON: Michelow (1963); **ELSYMS**: From CHEVRON; **BISTRO**: Schiffman (1962); **BISAR**: From BISTRO

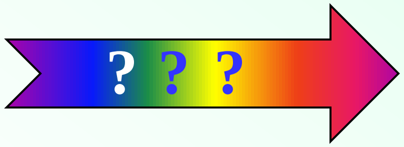


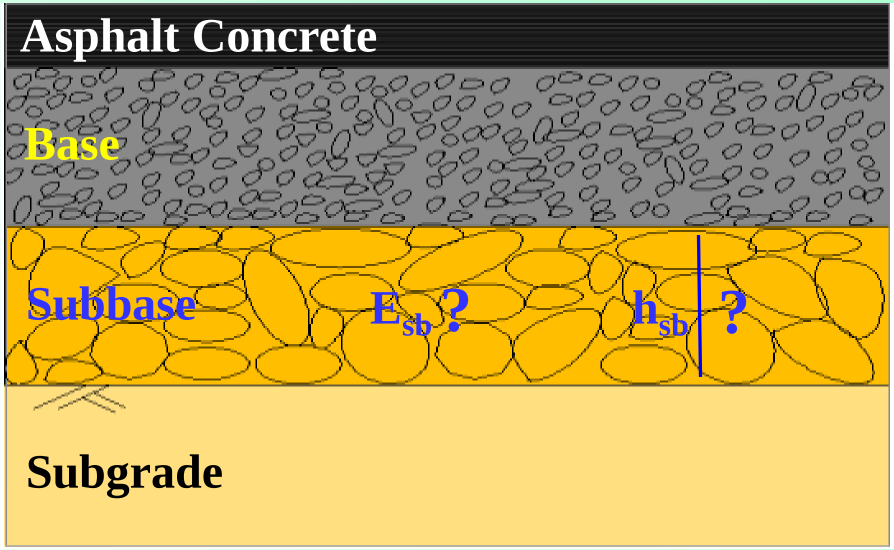


Schematic of Deflection Basin, and Loading, Sensors Configuration

FWD=Falling weight deflectometer

BackGenetic3D **for Backcalculation of Layered Moduli**





Pan (1989): PEPI; Pan (1997): AMM; Pan & Han (2005): IJSS; Pan et al. (2006): CMES

Problem Description

- Equilibrium equations

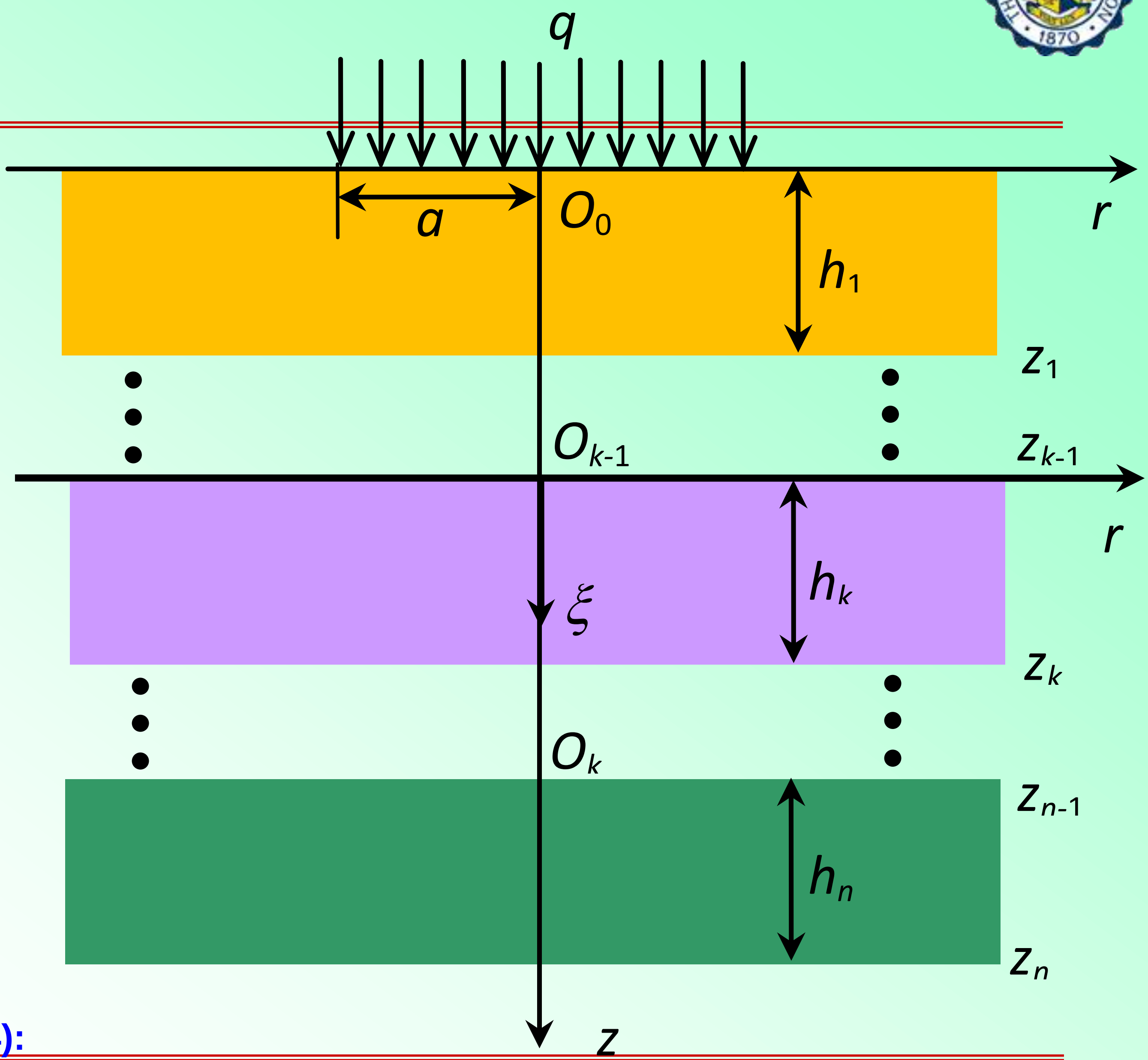
$$\sigma_{ji,j} + f_i = 0$$

- Constitutive relations

$$\sigma_{ij} = C_{ijkl}\gamma_{kl}$$

- Displacement-strain relations

$$\gamma_{ij} = 0.5(u_{i,j} + u_{j,i})$$



Xiao, HT and Yue, ZQ (2014): Fracture Mechanics in Layered and Graded Solids

Systems of Vector Functions

Direct expansion of any vector vs. scalar transform

- Cylindrical system of vector functions

$$\boldsymbol{L}(r,\theta;\lambda,m)=\boldsymbol{e}_z S(r,\theta;\lambda,m)$$

$$\boldsymbol{M}(r,\theta;\lambda,m)=(\boldsymbol{e}_r\frac{\partial}{\partial r}+\boldsymbol{e}_\theta\frac{\partial}{r\partial\theta})S(r,\theta;\lambda,m)$$

$$S(r,\theta;\lambda,m)=\frac{1}{\sqrt{2\pi}}J_m(\lambda r)e^{im\theta}$$

$$\boldsymbol{N}(r,\theta;\lambda,m)=(\boldsymbol{e}_r\frac{\partial}{r\partial\theta}-\boldsymbol{e}_\theta\frac{\partial}{\partial r})S(r,\theta;\lambda,m)$$

- Solutions in terms of cylindrical system of vector functions

$$\boldsymbol{u}(r,\theta,z)=\sum_m\int_0^{+\infty}[U_L(z)\boldsymbol{L}(r,\theta)+U_M(z)\boldsymbol{M}(r,\theta)+U_N(z)\boldsymbol{N}(r,\theta)]\lambda d\lambda$$

$$\boldsymbol{t}(r,\theta,z)\equiv\sigma_{rz}\boldsymbol{e}_r+\sigma_{\theta z}\boldsymbol{e}_\theta+\sigma_{zz}\boldsymbol{e}_z$$

$$=\sum_m\int_0^{+\infty}[T_L(z)\boldsymbol{L}(r,\theta)+T_M(z)\boldsymbol{M}(r,\theta)+T_N(z)\boldsymbol{N}(r,\theta)]\lambda d\lambda$$

○ quantities in physical domain ○ quantities in vector-function domain

Pan (Phys. Earth Planet. Inter.,1989a,b)

Two Independent Sub-Problems

Two sub-problems with clear/direct physical connection

- **N-Type** Solution

$$[\boldsymbol{E}^N]_{,z} = \lambda \begin{bmatrix} 0 & 1/C_{44}^0 \\ 1/C_{66}^0 & 0 \end{bmatrix} [\boldsymbol{E}^N] - \begin{bmatrix} 0 \\ F_N/\lambda \end{bmatrix} \qquad [\boldsymbol{E}^N(z)] = \begin{bmatrix} U_N(z) \\ T_N(z)/\lambda \end{bmatrix}$$

$$\Rightarrow \quad [\boldsymbol{E}^N] = [\boldsymbol{Z}^N(z)][\boldsymbol{K}^N]$$

- **LM-Type** Solution

$$[\boldsymbol{E}] = [U_L, \lambda U_M, T_L/\lambda, T_M]^t$$

$$[\boldsymbol{E}]_{,z} = \lambda[\boldsymbol{W}][\boldsymbol{E}] + [\boldsymbol{F}]$$

$$[\boldsymbol{F}] = [0, 0, -F_L/\lambda, -F_M]^t$$

$$\Rightarrow \quad [\boldsymbol{E}(z)] = [\boldsymbol{b}]e^{\lambda v z}$$

Eigensystem of Equations

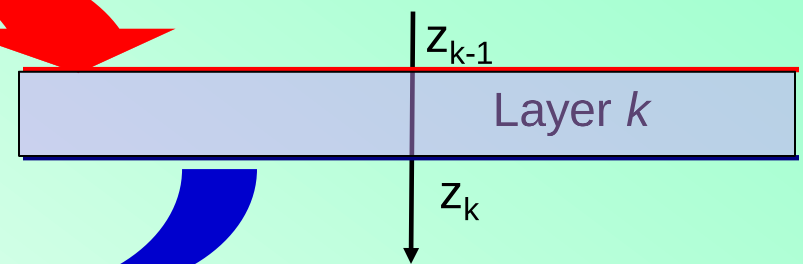
$$\{[\boldsymbol{W}] - v[\boldsymbol{I}]\}[\boldsymbol{b}] = 0$$

Propagator Matrix Method (PMM)

Without limitation for layer number, excellent for FGM

- Rlating $\boldsymbol{E}^N$ and $\boldsymbol{E}$ on the upper interface with those on the lower interface of layer k in transformed domain

LM-type $[\boldsymbol{E}(z_{k-1})]=[\boldsymbol{a}][\boldsymbol{E}(z_k)]$



N-type

$$[\boldsymbol{E}^N(z_{k-1})]=[\boldsymbol{a}^N(z)][\boldsymbol{E}^N(z_k)]$$

Propagator matrices:

$$[\boldsymbol{a}]=[\boldsymbol{B}]\left\langle e^{-\lambda v_k^* h_k}\right\rangle[\boldsymbol{B}]^{-1}$$

$$[\boldsymbol{a}^N]=[\boldsymbol{B}^N]\left\langle e^{-\lambda v_k^N h_k}\right\rangle[\boldsymbol{B}^N]^{-1}$$

Basic Properties of Matrix *a*

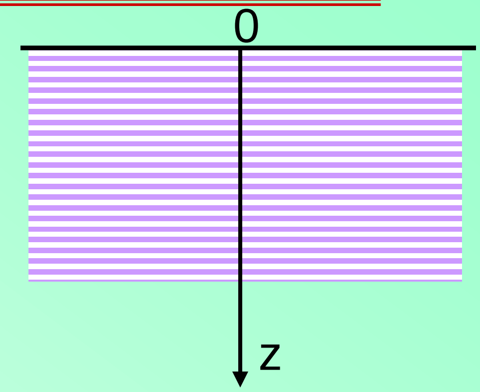
$$\boldsymbol{a}(0)=\begin{bmatrix}\boldsymbol{I} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{I}\end{bmatrix}$$

$$\boldsymbol{a}(z_3-z_1)=\boldsymbol{a}(z_3-z_2)\boldsymbol{a}(z_2-z_1)$$

$$\boldsymbol{a}(z_3-z_1)=\boldsymbol{a}^{-1}(z_1-z_3)$$

Solution in Transformed Domain

- Propagating from the bottom to top of the layered half space, we have



Unknowns

Traction B.C.

$$\begin{bmatrix} U_L(0) \\ \lambda U_M(0) \\ T_L(0)/\lambda \\ T_M(0) \end{bmatrix} = [\boldsymbol{M}] \begin{bmatrix} 0 \\ 0 \\ k_1 \\ k_2 \end{bmatrix}$$

Unknown far-field coefficients

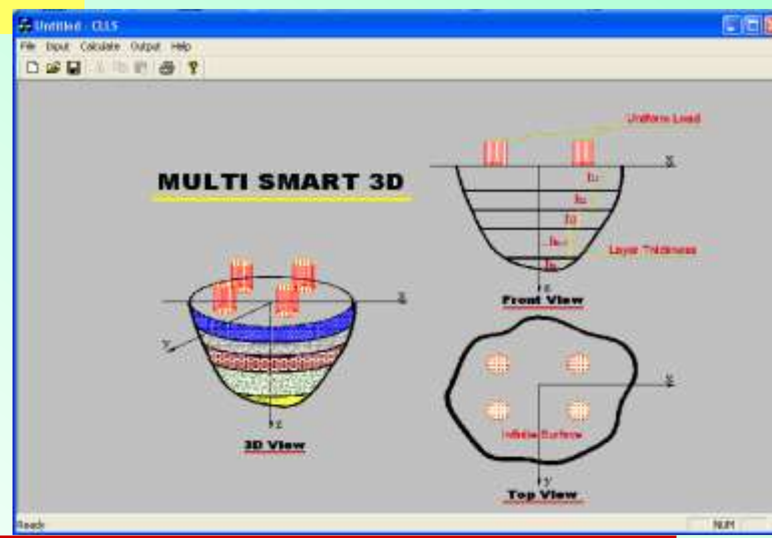
$$[\boldsymbol{M}] = [\boldsymbol{a}_1][\boldsymbol{a}_2] --- [\boldsymbol{a}_{p-1}][\boldsymbol{a}_p][\boldsymbol{Z}_p(H)]$$

Numerical Integral

- From transformed domain back to physical domain
- Infinite integrals involve Bessel functions (Inverse Hankel transformation)
- Adaptive Gauss quadrature integration (*Lucas, 1995; Pan & Han JEM 2004, IJSS 2005*)

$$\int_0^{+\infty} f(\lambda,z)J_m(\lambda r)d\lambda = \sum_{n=1}^{N} \int_{\lambda_n}^{\lambda_{n+1}} f(\lambda,z)J_m(\lambda r)d\lambda.$$

MultiSmart3D



World group in Dropbox with the program, manual, related articles, etc

Special Features of MultiSmart3D

Email me to join our Dropbox group

Layered pavement (Green, ODOT); Rock mechanics (Liao, Wang, NCTU); Layered Earth (Bevis, OSU)

1). ***Propagator matrix method*** is introduced so that one needs only to solve two 2×2 systems of linearly algebraic equations in the transformed domain, no matter how many layers we have in the layered structure!

2). ***Cylindrical system of vector functions*** is introduced so that the axisymmetric deformation can be exactly separated from the other part of the deformation!

3). ***Adaptive Gauss quadrature*** is utilized along with an acceleration approach for fast and accurate calculation of the integration!

4). ***Arbitrary observation point*** that can be at any location, far or near; immediately below or above any interface!

5). **Arbitrary interface** that can be assumed to have any different shear spring constant! (**Not for free**)

6). **Arbitrary surface loading/geometry with super-fast calculation!** (**Not for free**)

7). The research group has more than **30 years experience on the layered structure modeling and published more than 200 peer-reviewed journal papers in this specific area**!

US Patent 7627428 (12/01/09); Pan et al. (2007, GJI88, 90); …

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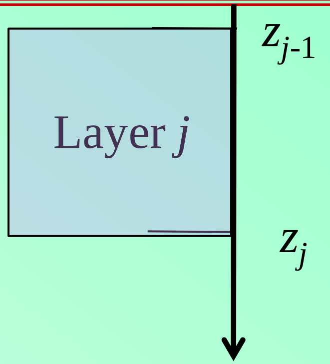
Stiffness Matrix Method (SMM)



$$\begin{bmatrix} \boldsymbol{T}_{j-1} \\ \boldsymbol{T}_j \end{bmatrix} = [\boldsymbol{K}_j] \begin{bmatrix} \boldsymbol{U}_{j-1} \\ \boldsymbol{U}_j \end{bmatrix}$$

vs. PMM

$$\begin{bmatrix} \boldsymbol{U}_j \\ \boldsymbol{T}_j \end{bmatrix} = [\boldsymbol{P}(z_j - z_{j-1})] \begin{bmatrix} \boldsymbol{U}_{j-1} \\ \boldsymbol{T}_{j-1} \end{bmatrix}$$

$$[\boldsymbol{K}_j] \equiv \begin{bmatrix} \boldsymbol{K}_{11}^j & \boldsymbol{K}_{12}^j \\ \boldsymbol{K}_{21}^j & \boldsymbol{K}_{22}^j \end{bmatrix}$$

$$\left[\boldsymbol{K}_j\right] = \begin{bmatrix} \boldsymbol{B} & \bar{\boldsymbol{B}} < \mathrm{e}^{\bar{s}^* \lambda h_j} > \\ \boldsymbol{B} < \mathrm{e}^{-s^* \lambda h_j} > & \bar{\boldsymbol{B}} \end{bmatrix} \begin{bmatrix} \boldsymbol{A} & \bar{\boldsymbol{A}} < \mathrm{e}^{\bar{s}^* \lambda h_j} > \\ \boldsymbol{A} < \mathrm{e}^{-s^* \lambda h_j} > & \bar{\boldsymbol{A}} \end{bmatrix}^{-1}$$

Precision Integration Method (PIM)

Zhong WX. Duality System in Applied Mechanics and Optimal Control. Kluwer Academic Publisher: Boston, 2004

$$\frac{d[\boldsymbol{V}]}{dz} = [\boldsymbol{H}][\boldsymbol{V}]$$

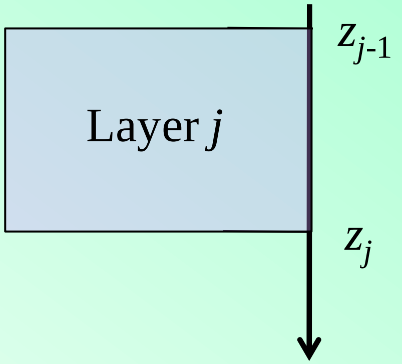
For a set of given ODEs

$$\frac{d}{dz}\begin{bmatrix}\boldsymbol{U}\\ \boldsymbol{T}\end{bmatrix} = \begin{bmatrix}\boldsymbol{A} & \boldsymbol{D}\\ \boldsymbol{B} & \boldsymbol{C}\end{bmatrix}\begin{bmatrix}\boldsymbol{U}\\ \boldsymbol{T}\end{bmatrix}$$

$$[\boldsymbol{H}] = \lambda \cdot \begin{bmatrix} 0 & c_{13}/c_{33} & 1/c_{33} & 0 \\ -1 & 0 & 0 & 1/c_{44} \\ 0 & 0 & 0 & 1 \\ 0 & c_{11} - c_{13}^2/c_{33} & -c_{13}/c_{33} & 0 \end{bmatrix}$$

It's solution can be expressed as

$$\begin{bmatrix}\boldsymbol{U}_j\\ \boldsymbol{T}_j\end{bmatrix} = \exp\left\{\begin{bmatrix}\boldsymbol{A} & \boldsymbol{D}\\ \boldsymbol{B} & \boldsymbol{C}\end{bmatrix}(z_j - z_{j-1})\right\}\begin{bmatrix}\boldsymbol{U}_{j-1}\\ \boldsymbol{T}_{j-1}\end{bmatrix}$$



Alternatively as

$$\boldsymbol{U}_j = \boldsymbol{F}\boldsymbol{U}_{j-1} - \boldsymbol{G}\boldsymbol{T}_j$$
$$\boldsymbol{T}_{j-1} = \boldsymbol{Q}\boldsymbol{U}_{j-1} + \boldsymbol{E}\boldsymbol{T}_j$$

vs. PMM $\begin{bmatrix}\boldsymbol{U}_j\\ \boldsymbol{T}_j\end{bmatrix} = [\boldsymbol{P}_j]\begin{bmatrix}\boldsymbol{U}_{j-1}\\ \boldsymbol{T}_{j-1}\end{bmatrix}$

vs. SMM $\begin{bmatrix}\boldsymbol{T}_{j-1}\\ \boldsymbol{T}_j\end{bmatrix} = [\boldsymbol{K}_j]\begin{bmatrix}\boldsymbol{U}_{j-1}\\ \boldsymbol{U}_j\end{bmatrix}$

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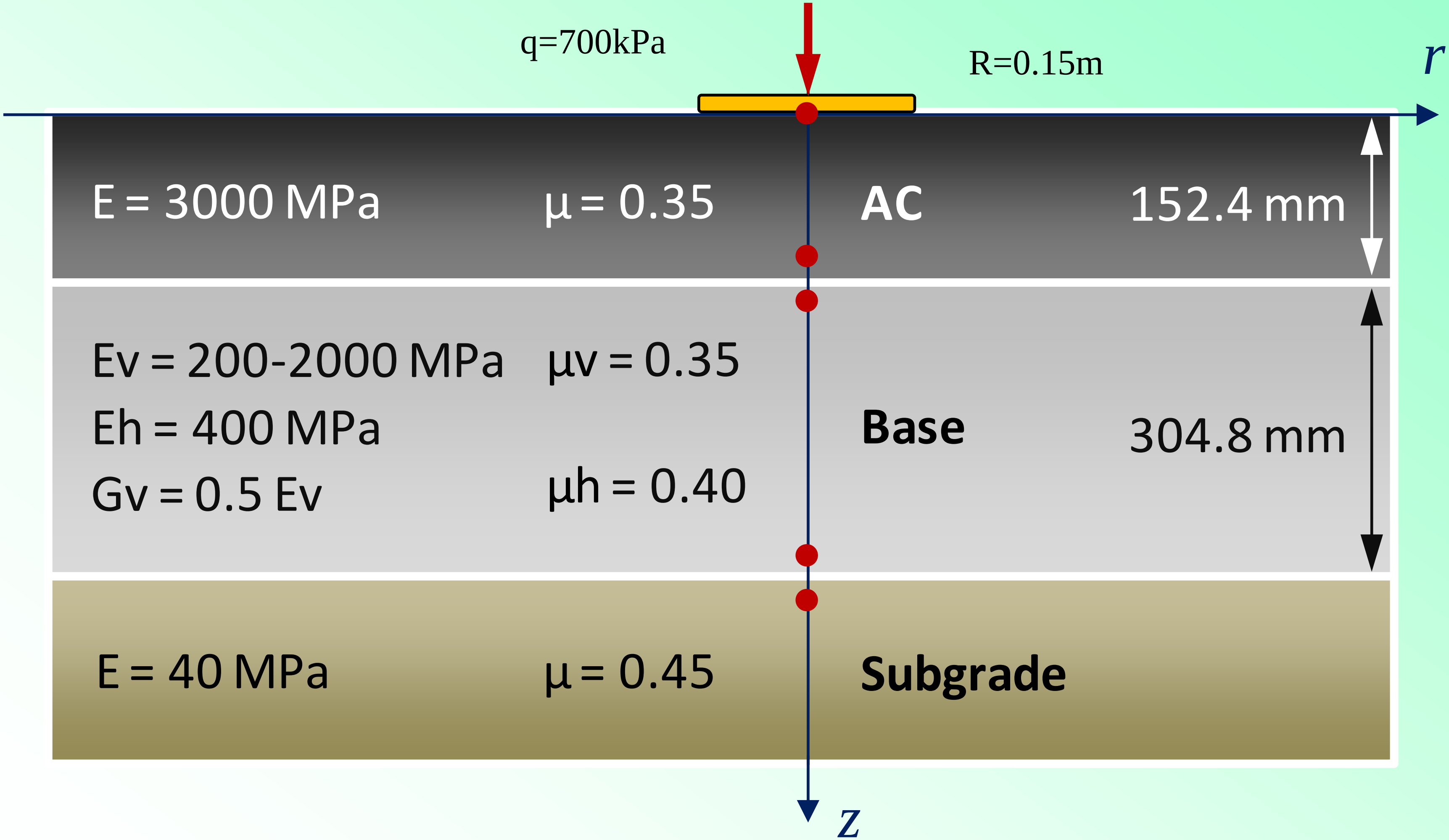
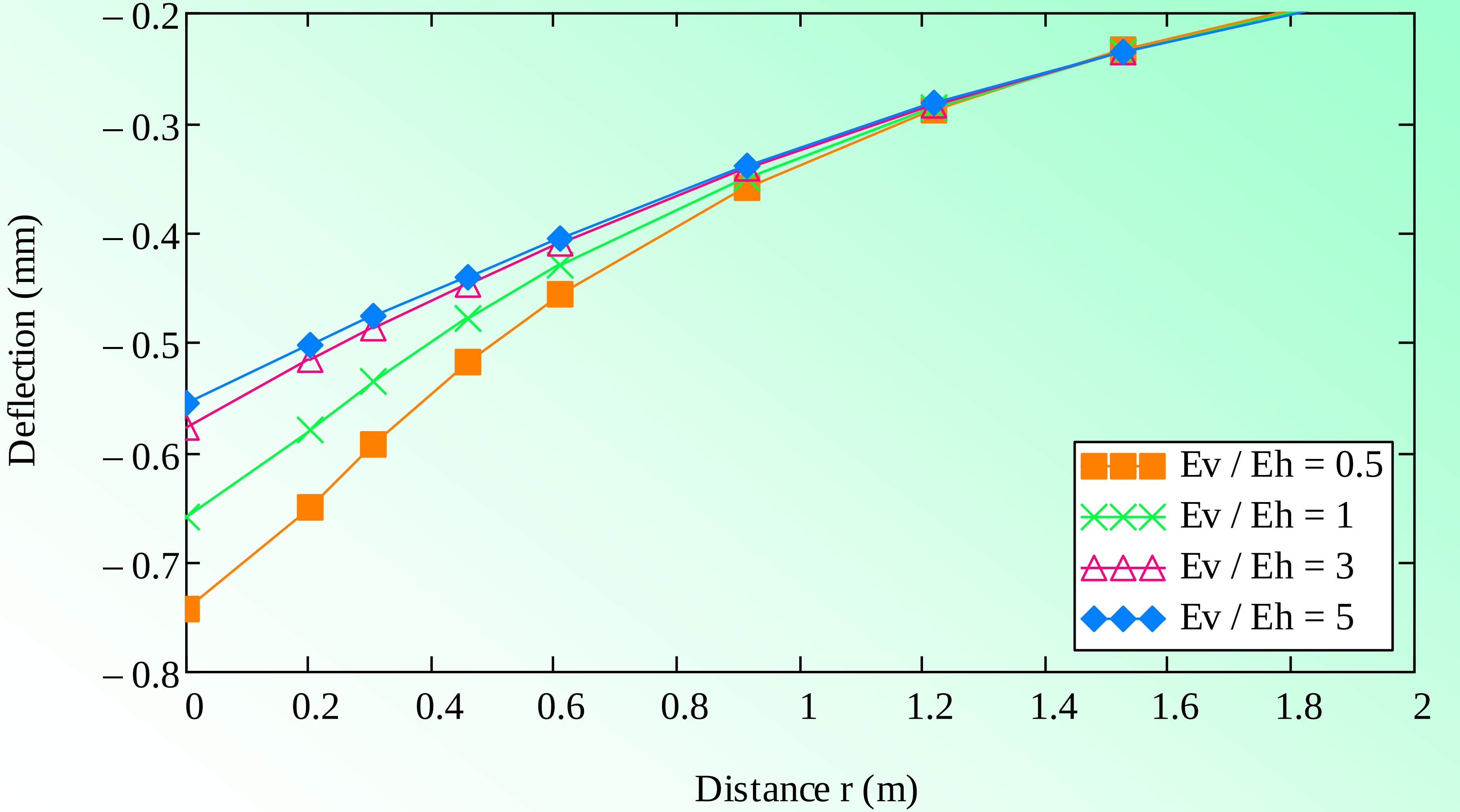
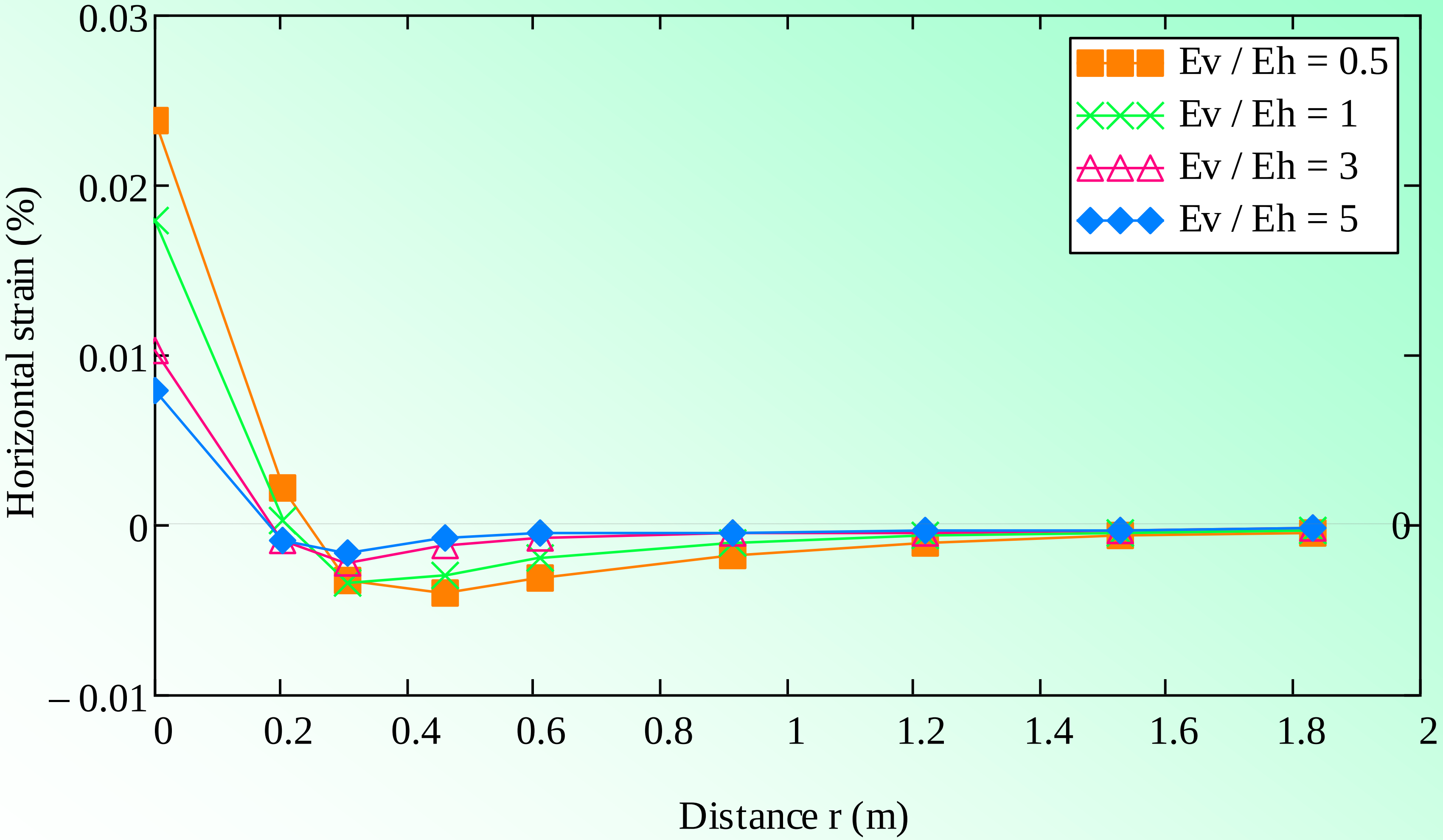
Effect of TI on Pavement Response



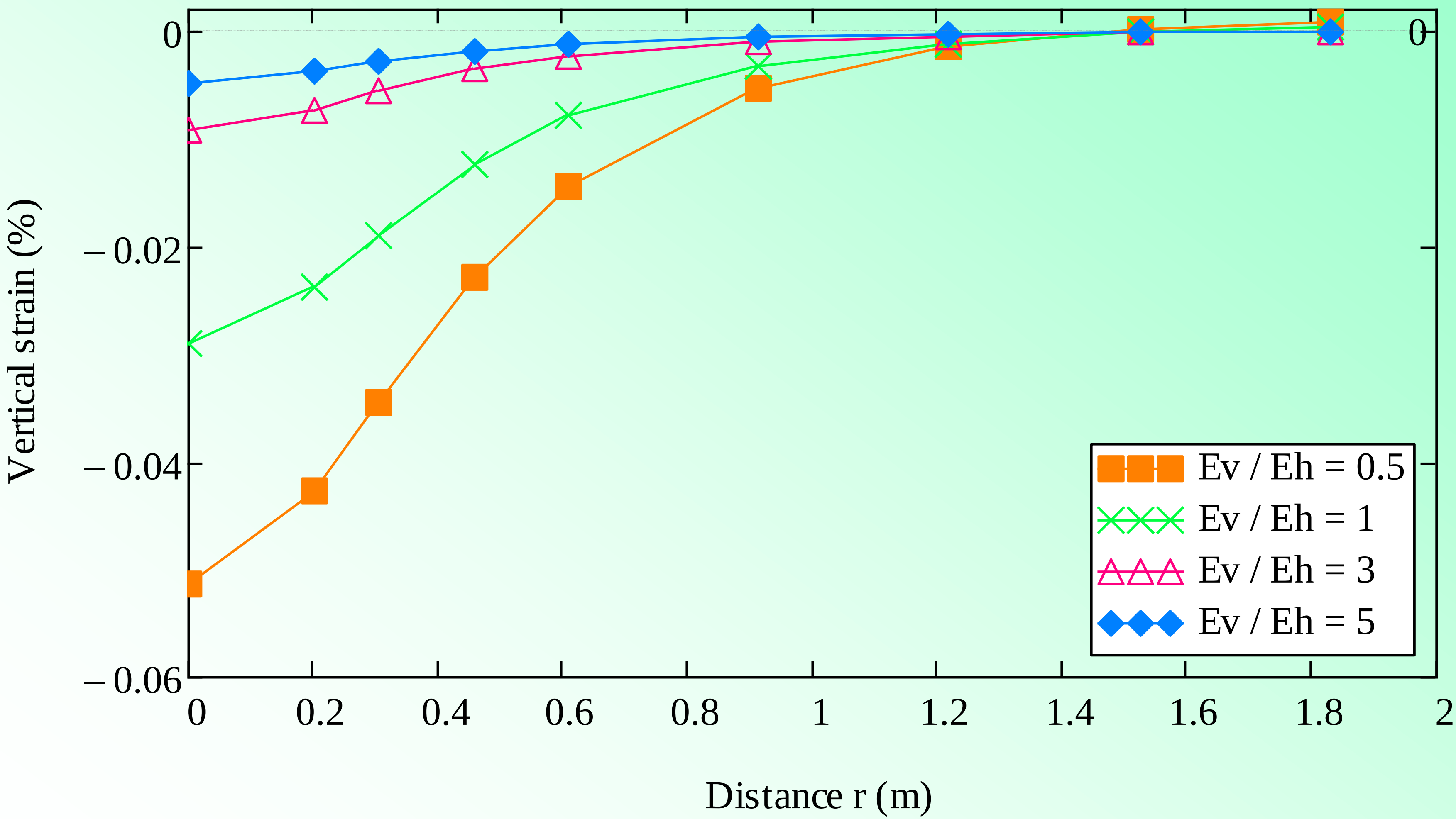
Deflection on the Surface



Tensile Strain on the Bottom of AC Layer



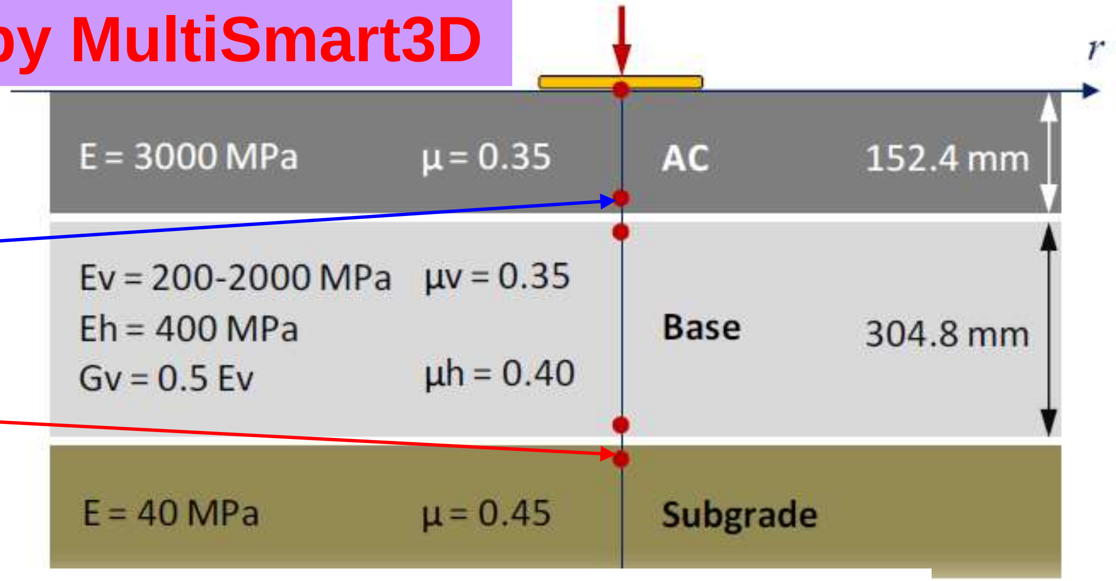
Compressive Strain on the Top of Subgrade

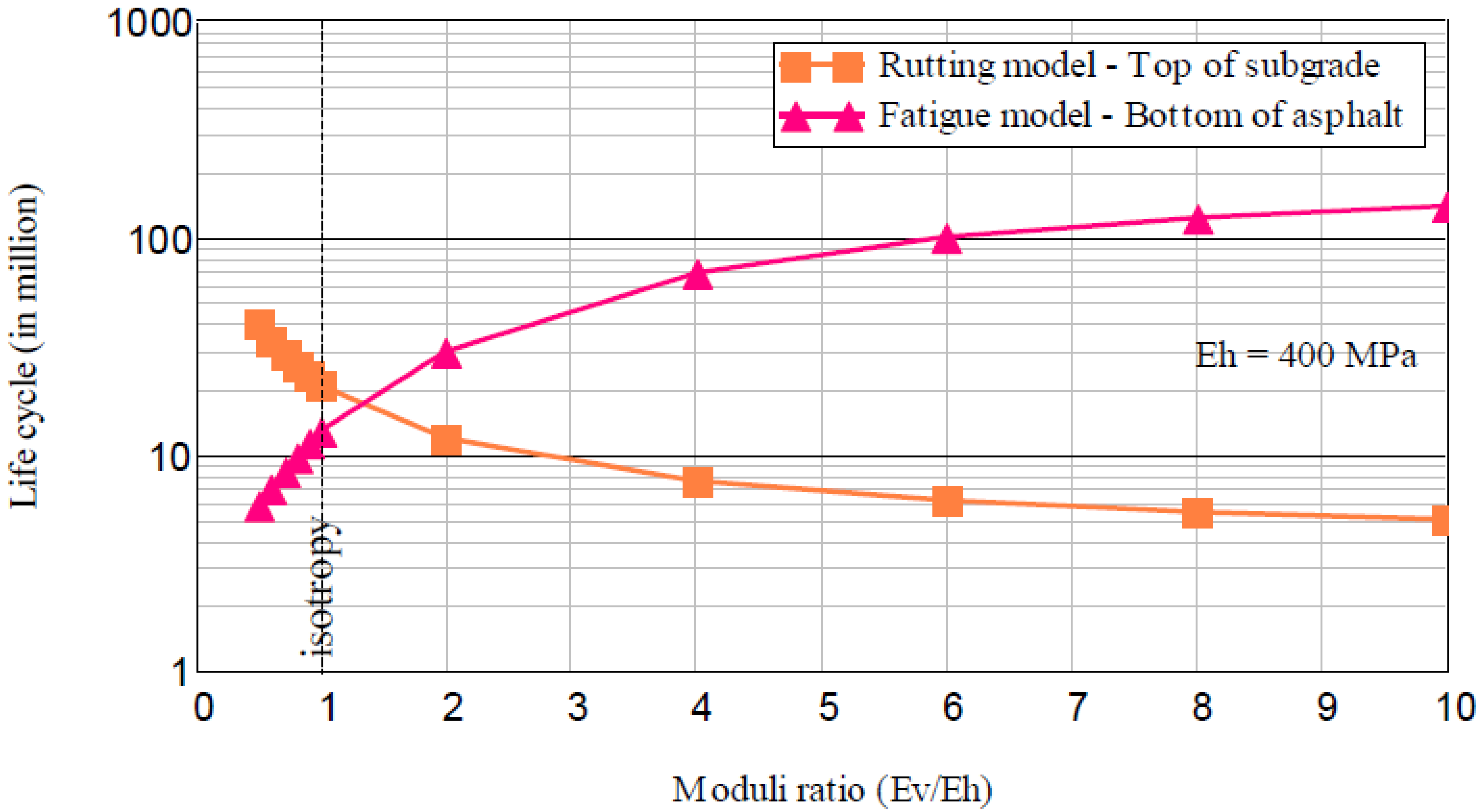


Forward Calculation by MultiSmart3D

$$N_f = (1.660\times10^{-10})(1/\varepsilon_r)^{4.32}$$

$$N_r = f_1(\varepsilon_v)^{-f_2}$$





Cai et al. 2015
Comput. Geotech

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SID-Pavement Made of Four Layers (Model 3)

NO.	Layer	Parameter	Seed modulus (MPa)	Inverse modulus (MPa)	Error (%)
1	AC	E_{ac}	30000	1499.992	-0.001
	Base	E_{bh}	3000	500.509	0.102
		E_{bv}	40000	799.901	-0.012
	Subbase	E_{sbh}	3000	299.907	-0.031
		E_{sbv}	40000	500.149	0.030
	Subgrade	E_{sg}	300	30	0.000
2	AC	E_{ac}	10000	1499.993	0.000
	Base	E_{bh}	10000	500.541	0.108
		E_{bv}	20000	799.9	-0.013
	Subbase	E_{sbh}	10000	299.903	-0.032
		E_{sbv}	20000	500.146	0.029
	Subgrade	E_{sg}	10000	30	0.000
3	AC	E_{ac}	300	1499.993	0.000
	Base	E_{bh}	3000	500.534	0.107
		E_{bv}	3000	799.901	-0.012
	Subbase	E_{sbh}	3000	299.906	-0.031
		E_{sbv}	3000	500.146	0.029
	Subgrade	E_{sg}	30000	30	0.000

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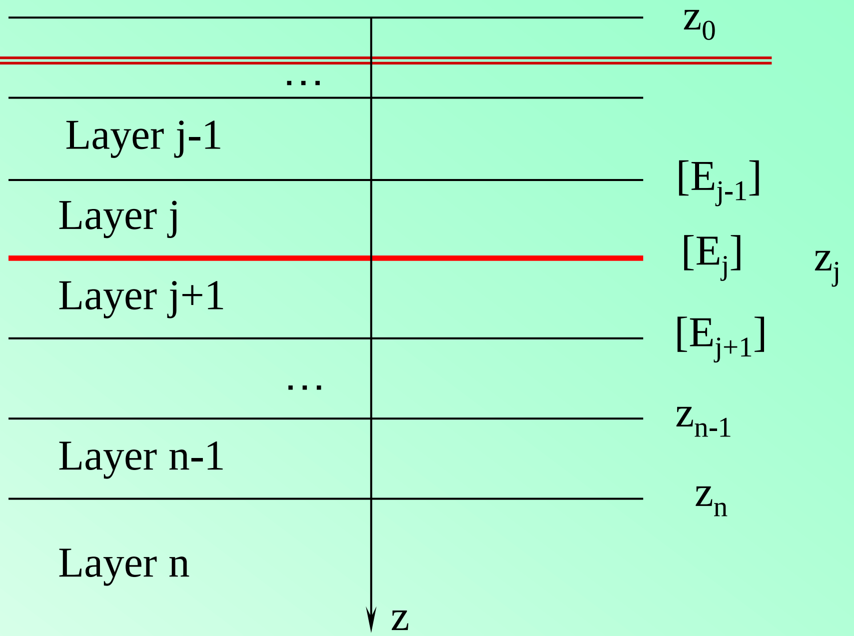
Forward and Backcalculation of Pavements with Bonding Interface

Bonding Condition



$$\tau = k(\Delta U) \quad k = 1/\alpha$$

$$\sigma_{xz}(x,y,z_j-0) = \sigma_{xz}(x,y,z_j+0)$$

$$\sigma_{yz}(x,y,z_j-0) = \sigma_{yz}(x,y,z_j+0)$$

$$\sigma_{zz}(x,y,z_j-0) = \sigma_{zz}(x,y,z_j+0)$$

$$u_x(x,y,z_j-0) - u_x(x,y,z_j+0) = \alpha_x^{(j)}\sigma_{xz}(x,y,z_j-0)$$

$$u_y(x,y,z_j-0) - u_y(x,y,z_j+0) = \alpha_x^{(j)}\sigma_{yz}(x,y,z_j-0)$$

$$u_z(x,y,z_j-0) - u_z(x,y,z_j+0) = \alpha_z^{(j)}\sigma_{zz}(x,y,z_j-0)$$

Solutions in terms of Propagator

$$[E(z_0)] = [a_1][a_2]\cdots[a_N][E(z_N)]$$

Without bonding interface

$$[E(z)] = [a_{z_k - h}][a_{z_{k+1}}]\cdots[a_N][E(z_N)]$$

$$[E(z_0)] = [a_1][a_2]\cdots[InterM][a_j]\cdots[a_N][E(z_N)]$$

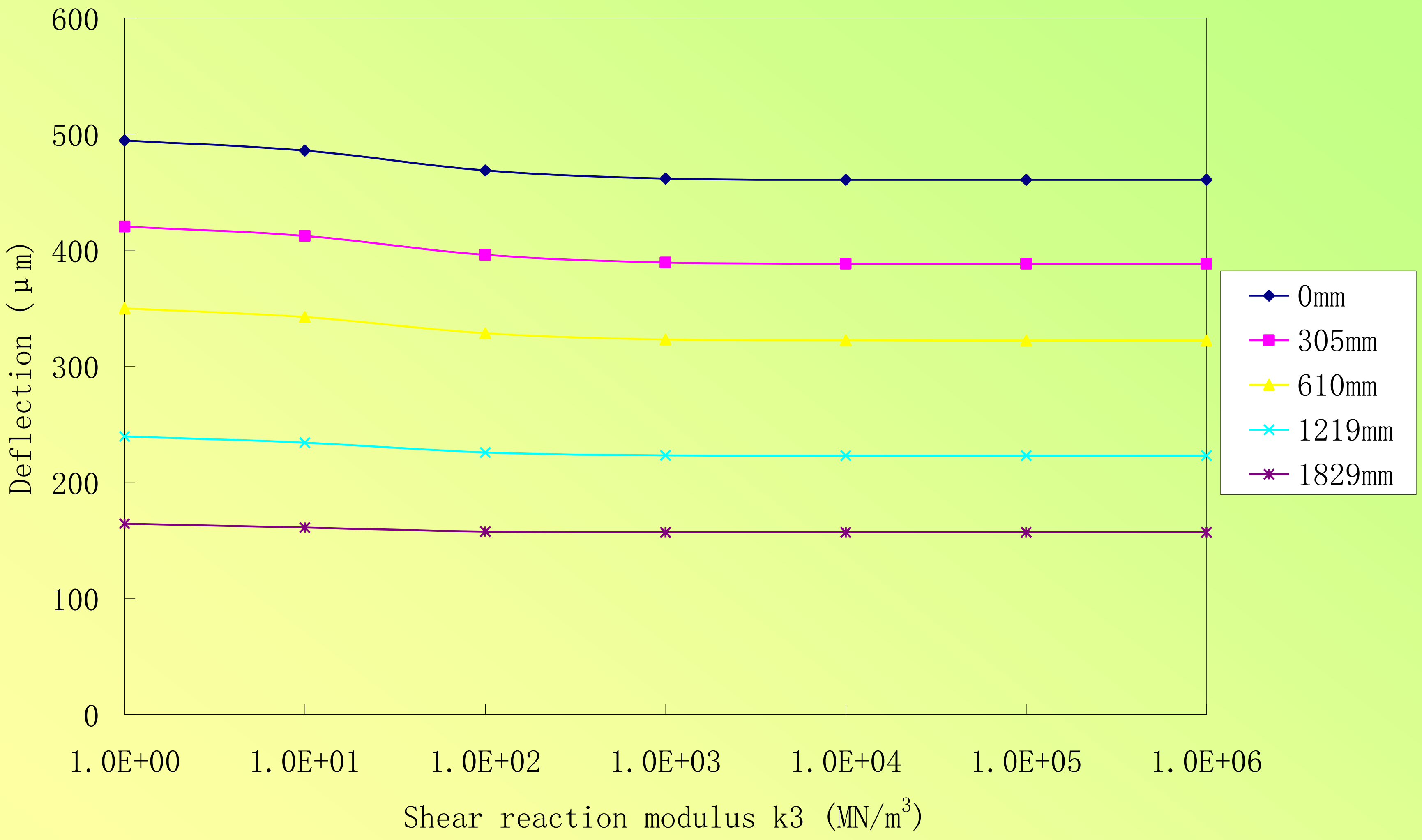
With bonding interface

$$[E(z)] = [a_{z_k - h}][a_{z_{k+1}}]\cdots[InterM][a_j]\cdots[a_N][E(z_N)]$$

Effect of Bonding Condition on Pavement Response

<table>
<tr><th>Layer</th><th>Modulus(MPa)</th><th>Poisson Ratio</th><th>Thickness(mm)</th><th>Interface</th></tr>
<tr><td rowspan="2">Asphalt</td><td rowspan="2">5000</td><td rowspan="2">0.35</td><td rowspan="2">150</td><td></td></tr>
<tr><td rowspan="2">k1</td></tr>
<tr><td rowspan="2">Base</td><td rowspan="2">1200</td><td rowspan="2">0.40</td><td rowspan="2">180</td></tr>
<tr><td rowspan="2">k2</td></tr>
<tr><td rowspan="2">Subbase</td><td rowspan="2">300</td><td rowspan="2">0.40</td><td rowspan="2">180</td></tr>
<tr><td rowspan="2">k3</td></tr>
<tr><td rowspan="2">Subgrade</td><td rowspan="2">50</td><td rowspan="2">0.45</td><td rowspan="2">infinity</td></tr>
<tr><td></td></tr>
</table>

Bonding k3 Between Subbase & Subgrade



SID-Backcalculation with k3

Pavement	Layer	Seed value	Real value	Backcalculated value	error(%)
1	E_{ac}	30000	5000	5000.018	0.000
	E_b	3000	1200	1199.989	-0.001
	E_{sb}	3000	300	300.007	0.002
	k_3	1	1000	999.091	-0.091
	E_{sg}	300	50	50	0.000
2	E_{ac}	300	5000	4999.968	-0.001
	E_b	3000	1200	1200.013	0.001
	E_{sb}	30000	300	299.996	-0.001
	k_3	1	1000	1000.015	0.001
	E_{sg}	3000	50	50	0.000
3	E_{ac}	10000	5000	5000.019	0.000
	E_b	10000	1200	1199.988	-0.001
	E_{sb}	10000	300	300.009	0.003
	k_3	100	1000	998.013	-0.199
	E_{sg}	10000	50	50	0.000

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Conclusions

- ***MultiSmart3D*** based on the cylindrical system of vector functions is robust for transforming a PDE into an ODE. Vector system has clear physical meanings.
- PMM,SMM and PIM are feasible methods for forward calculation of TI multilayered system with imperfect interface.
- SID is a powerful method for backcalculating both moduli/bonding conditions in TI system.

Thank you very much!!!
Don’t forget the wonderful Pan

Backups

Stiffness Matrix Method (SMM)

$$\frac{d[\boldsymbol{V}]}{dz}=[\boldsymbol{H}][\boldsymbol{V}]$$

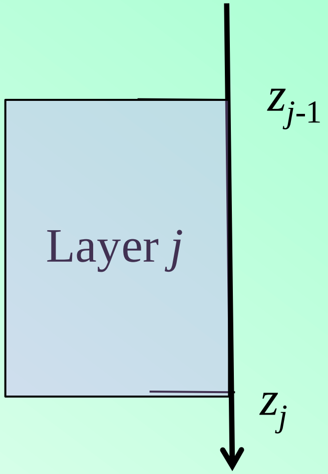
$$[\boldsymbol{V}]=[\boldsymbol{U}_L,\lambda\boldsymbol{U}_M,\boldsymbol{T}_L/\lambda,\boldsymbol{T}_M]^T$$

$$[\boldsymbol{H}]=\lambda\cdot\begin{bmatrix} 0 & c_{13}/c_{33} & 1/c_{33} & 0 \\ -1 & 0 & 0 & 1/c_{44} \\ 0 & 0 & 0 & 1 \\ 0 & c_{11}-c_{13}^2/c_{33} & -c_{13}/c_{33} & 0 \end{bmatrix}$$

Solutions of the ODEs

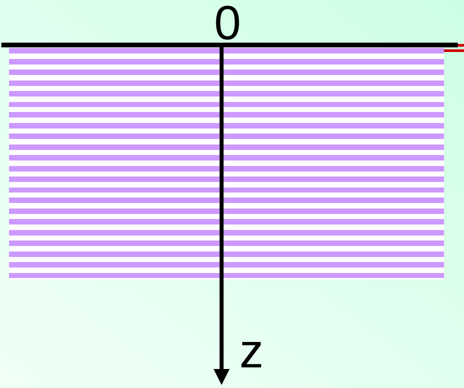
$$\begin{bmatrix}\boldsymbol{U}(z)\\ \boldsymbol{T}(z)\end{bmatrix} = \begin{bmatrix}\boldsymbol{A} & \bar{\boldsymbol{A}}\\ \boldsymbol{B} & \bar{\boldsymbol{B}}\end{bmatrix}\begin{bmatrix}< \mathrm{e}^{s^*\lambda(z-z_j)} > & \mathbf{0}\\ \mathbf{0} & < \mathrm{e}^{-s^*\lambda(z-z_{j-1})} >\end{bmatrix}\begin{bmatrix}\boldsymbol{K}^+\\ \boldsymbol{K}^-\end{bmatrix}$$

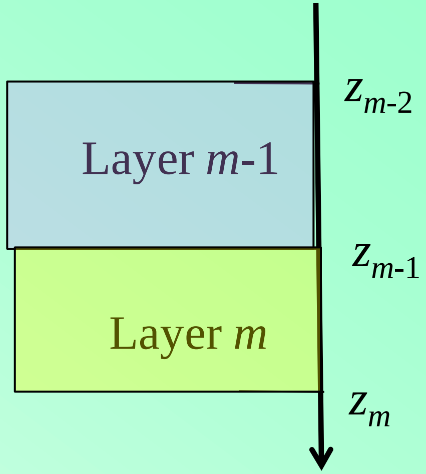
$$\begin{bmatrix}\boldsymbol{U}(z)\\ \boldsymbol{T}(z)\end{bmatrix} \equiv \begin{bmatrix}U_L\\ \lambda U_M\\ T_L/\lambda\\ T_M\end{bmatrix}$$



$$< \mathrm{e}^{-s^*\lambda z} > = diag[\mathrm{e}^{-s_1\lambda z}, \mathrm{e}^{-s_2\lambda z}, \mathrm{e}^{-s_3\lambda z}, \mathrm{e}^{-s_4\lambda z}]$$

Recursive Relations Between Adjacent Layers





$$\begin{bmatrix} \boldsymbol{T}_{m-2} \\ \boldsymbol{T}_{m} \end{bmatrix} = [\boldsymbol{K}_{m-2:m}] \begin{bmatrix} \boldsymbol{U}_{m-2} \\ \boldsymbol{U}_{m} \end{bmatrix}$$

$$[\boldsymbol{K}_{m-2:m}] = \begin{bmatrix} \boldsymbol{K}_{11}^{m-1} - \left(\boldsymbol{K}_{12}^{m-1}\right)\left(\boldsymbol{K}_{22}^{m-1} - \boldsymbol{K}_{11}^{m}\right)^{-1} \boldsymbol{K}_{21}^{m-1} & \left(\boldsymbol{K}_{12}^{m-1}\right)\left(\boldsymbol{K}_{22}^{m-1} - \boldsymbol{K}_{11}^{m}\right)^{-1} \boldsymbol{K}_{12}^{m} \\ -\boldsymbol{K}_{21}^{m}\left(\boldsymbol{K}_{22}^{m-1} - \boldsymbol{K}_{11}^{m}\right)^{-1} \boldsymbol{K}_{21}^{m-1} & \boldsymbol{K}_{21}^{m}\left(\boldsymbol{K}_{22}^{m-1} - \boldsymbol{K}_{11}^{m}\right)^{-1} \boldsymbol{K}_{12}^{m} + \boldsymbol{K}_{22}^{m} \end{bmatrix}$$

Finding Matrices G, F, E, Q…

$$\tau = z_j - z_{j-1}$$

$$d\boldsymbol{G} / d\tau = (\boldsymbol{GB} + \boldsymbol{A})\boldsymbol{G} - \boldsymbol{D} - \boldsymbol{GC}$$

$$d\boldsymbol{F} / d\tau = (\boldsymbol{GB} + \boldsymbol{A})\boldsymbol{F}$$

$$d\boldsymbol{E} / d\tau = \boldsymbol{EBG} - \boldsymbol{EC}$$

$$d\boldsymbol{Q} / d\tau = -\boldsymbol{EBF}$$

Expansion coefficients of ***G,F,E,Q***

$$\boldsymbol{G}(\tau) = \boldsymbol{g}_1\tau + \boldsymbol{g}_2\tau^2 + \boldsymbol{g}_3\tau^3 + \boldsymbol{g}_4\tau^4 + \cdots$$

$$\boldsymbol{Q}(\tau) = \boldsymbol{q}_1\tau + \boldsymbol{q}_2\tau^2 + \boldsymbol{q}_3\tau^3 + \boldsymbol{q}_4\tau^4 + \cdots$$

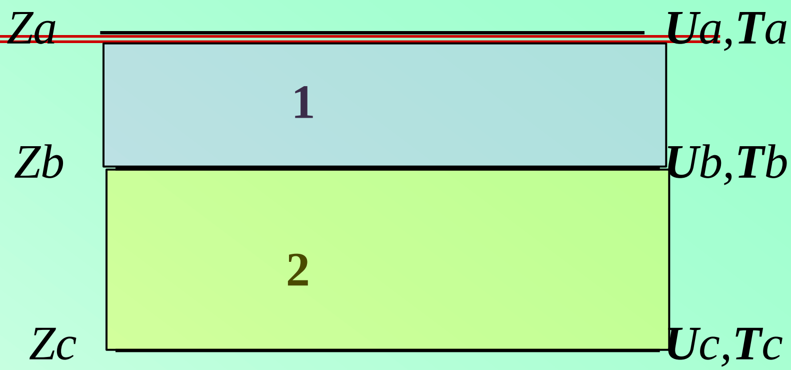
$$\boldsymbol{F}(\tau) = \boldsymbol{I} + \boldsymbol{f}_1\tau + \boldsymbol{f}_2\tau^2 + \boldsymbol{f}_3\tau^3 + \boldsymbol{f}_4\tau^4 + \cdots$$

$$\boldsymbol{E}(\tau) = \boldsymbol{I} + \boldsymbol{e}_1\tau + \boldsymbol{e}_2\tau^2 + \boldsymbol{e}_3\tau^3 + \boldsymbol{e}_4\tau^4 + \cdots$$

Recursive Relations For Multilayered Systems

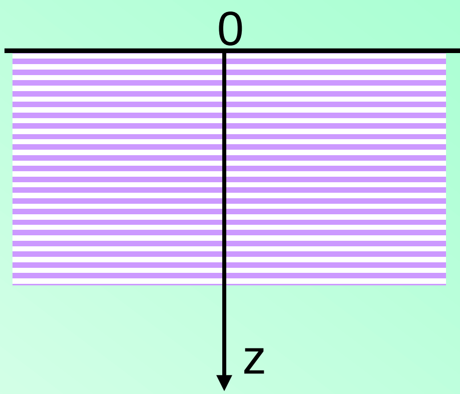
$$U_b = F_1 U_a - G_1 T_b$$

$$T_a = Q_1 U_a + F_1^t T_b$$



$$U_c = F_2 U_b - G_2 T_c$$

$$T_b = Q_2 U_b + F_2^t T_c$$



$$U_c = F_3 U_a - G_3 T_c$$

$$T_a = Q_3 U_a + F_3^t T_c$$

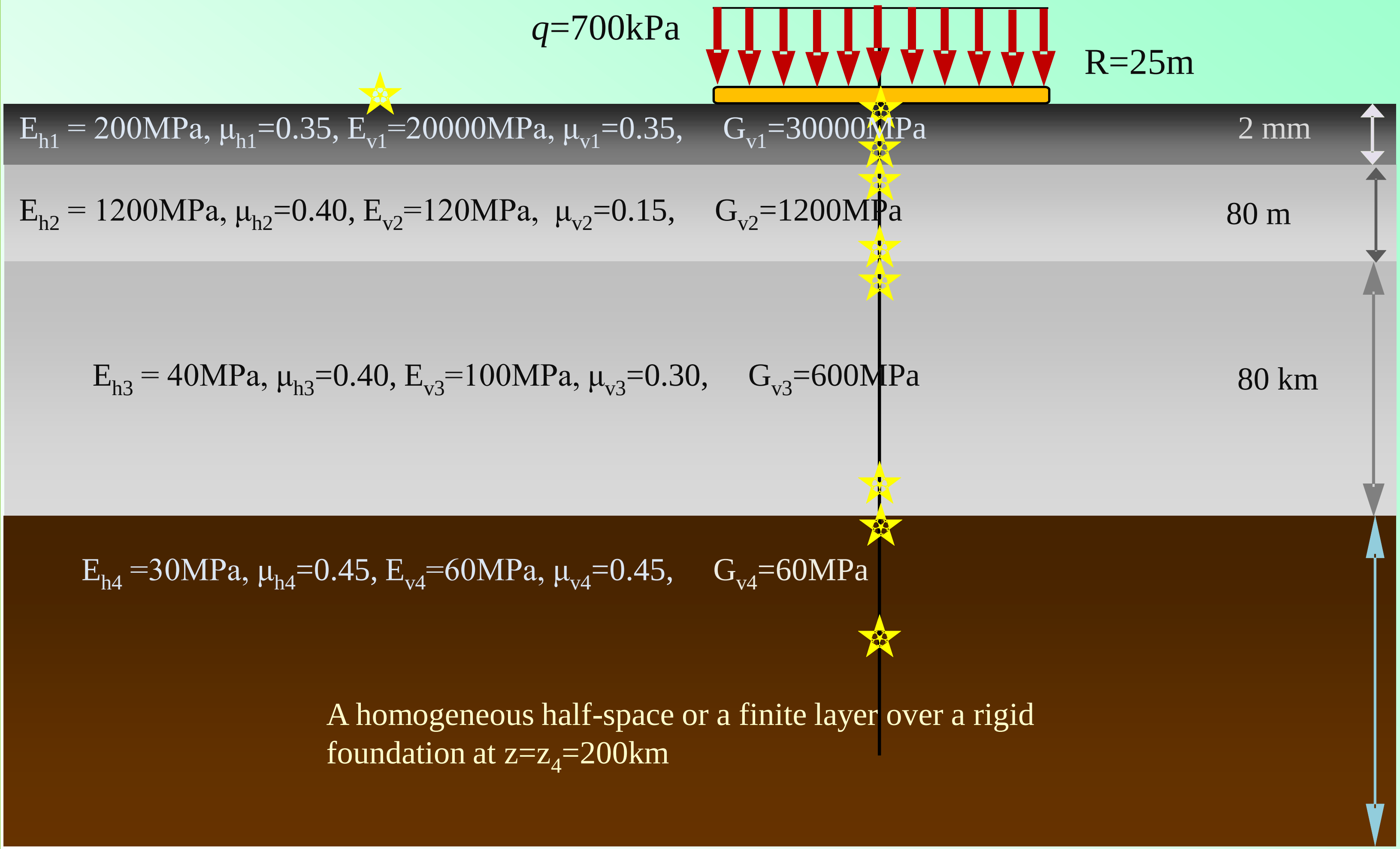
$$Q_3 = Q_1 + F_1^t (Q_2^{-1} + G_1)^{-1} F_1$$

$$F_3^t = F_1 (I + Q_2 G_1)^{-1} F_2^t$$

$$G_3 = G_2 + F_2 (G_1^{-1} + Q_2)^{-1} F_2^t$$

$$F_3 = F_2 (I + G_1 Q_2)^{-1} F_1$$

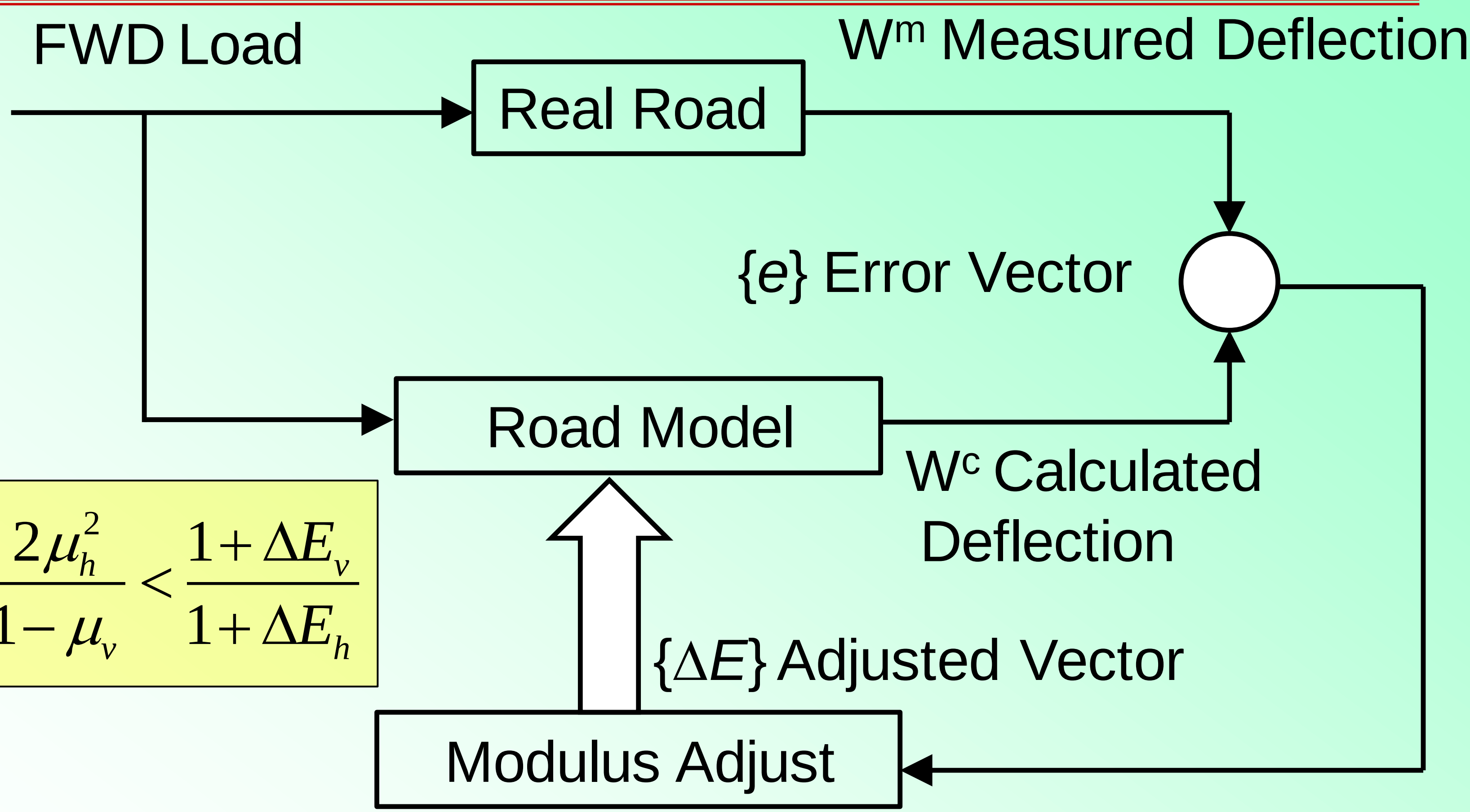
Example 2 (Model 2)



Displacements and stresses at different field points of the layered pavement Model 2 based on the PMM method (u_z in mm/1000 and $\sigma_{\theta\theta}$ in Pa).

Field point (m) (y=0)		Three layers over a half-space		Four layers over a rigid foundation at $z=z_4$=200km	
x	z	u_z	$\sigma_{\theta\theta}$	u_z	$\sigma_{\theta\theta}$
0	0	71976.976	-48662.12	71851.61	-48662.09
120	0	18113.922	-464.604	17993.20	-464.508
0	0.0019	71976.913	-48635.483	71851.549	-48635.452
0	0.0021	71976.836	-2041583.28	71851.473	-2041583.08
0	80.0019	14873.616	45180.920	14748.127	45181.12
0	80.0021	14873.594	893.6888	14784.106	893.6955
0	80080.0019	169.3834	0.0000	41.3067	0.0000
0	80080.0021	169.3834	0.0000	41.3067	0.0000
0	90000	160.5567	0.0000	34.6552	0.0000

Backcalculation of Pavement Properties



$$0 < \frac{E_h^i}{E_v^i} \cdot \frac{2\mu_h^2}{1-\mu_v} < \frac{1+\Delta E_v}{1+\Delta E_h}$$

System Identification (SID), Robert Lytton/Fuming Wang

Example One

P=700kPa, R=0.15m?

	Modulus (MPa)	Poisson's Ratio	Thickness (mm)	Surface location (mm)	Deflection (mm/1000)	
10 Layers	3500	0.35	150		**MS3D**	**BISAR**
	1000	0.35	50	0	360.724	360.448
	800	0.35	250	305	283.237	282.931
	500	0.4	180	609	239.488	239.177
	400	0.4	100	914	212.296	211.967
	200	0.4	250	1219	191.147	190.801
	150	0.45	320	1524	173.346	172.986
	100	0.45	180	1829	157.904	157.537
	80	0.45	100			
	40	0.45				

Example Two

Structure	Layer	Modulus(MPa)	Poisson	Thickness(m)	
	1	30000	0.40	0.50	
	2	300	0.40	0.80	
	3	30	0.40		
Load		Magnitude (kPa)	Radius(m)	Location(x)	Location(y)
	V	0.10	0.15	0	0
	H	693.56	0.15	0.00	0.00

MultiSmart3D

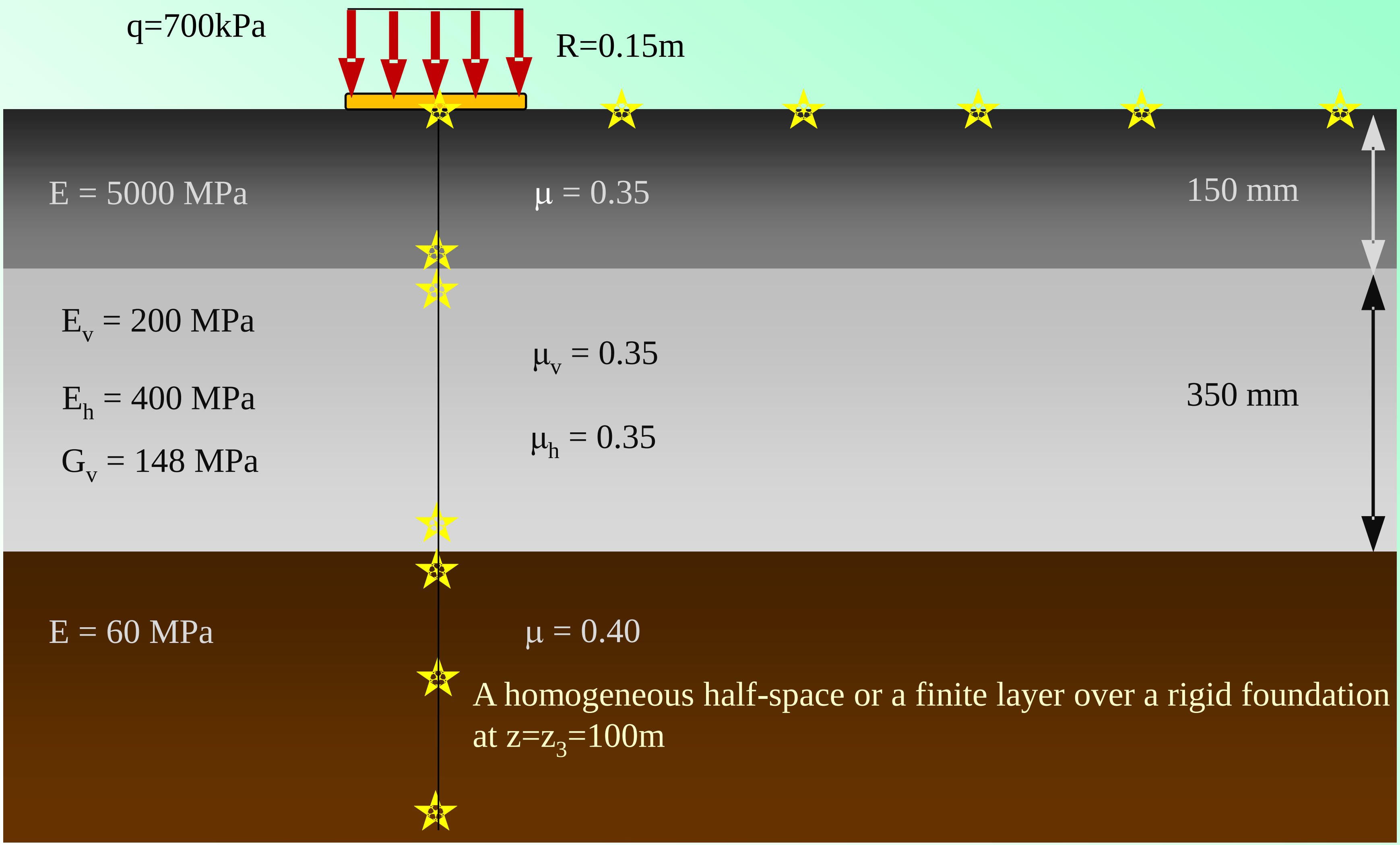
2016/12/21 other methods

Position			Displacement(μm)			Stress(MPa)		
x	y	z	x	y	z	xx	yy	zz
0	0	0	14.779	0.00	0.0251	-130.1	-130.1	-100.0
0.305	0	0	9.186	0.00	1.851	-287693.7	-36140.9	0.0
0.609	0	0	7.647	0.00	2.863	-106141.0	-27356.5	0.0
1.219	0	0	6.458	0.00	3.953	-45508.1	-16094.1	0.0
1.829	0	0	5.889	0.00	4.372	-26721.2	-10299.8	0.0
0	0	0.3	7.067	0.00	0.025	9.6	9.6	-20.0
0.2	0.2	0	9.025	0.3729	1.257	-165959.0	-95192.4	0.0
0.3	0.6	0.3	6.137	0.1995	1.487	-1874.0	-5358.0	-297.6

BISAR

Position			Displacement(μm)			Stress(MPa)		
x	**y**	**z**	**x**	**y**	**z**	**xx**	**yy**	**zz**
0	0	0	14.780	0	0.0251	-130.1	-130.1	-100.0
0.305	0	0	9.236	0	1.851	-287100.0	-36710.0	0.0
0.609	0	0	7.670	0	2.862	-105900.0	-27580.0	0.0
1.219	0	0	6.472	0	3.953	-44800.0	-16800.0	0.0
1.829	0	0	5.88	0	4.372	-26720.0	-10300.0	0.0
0	0	0.3	7.067	0	0.0247	9.6	9.6	-20.0
0.2	0.2	0	9.025	0.4264	1.257	-166000.0	-95170.0	0.0
0.3	0.6	0.3	6.123	0.2178	1.487	-1623.0	-5610.0	-297.5

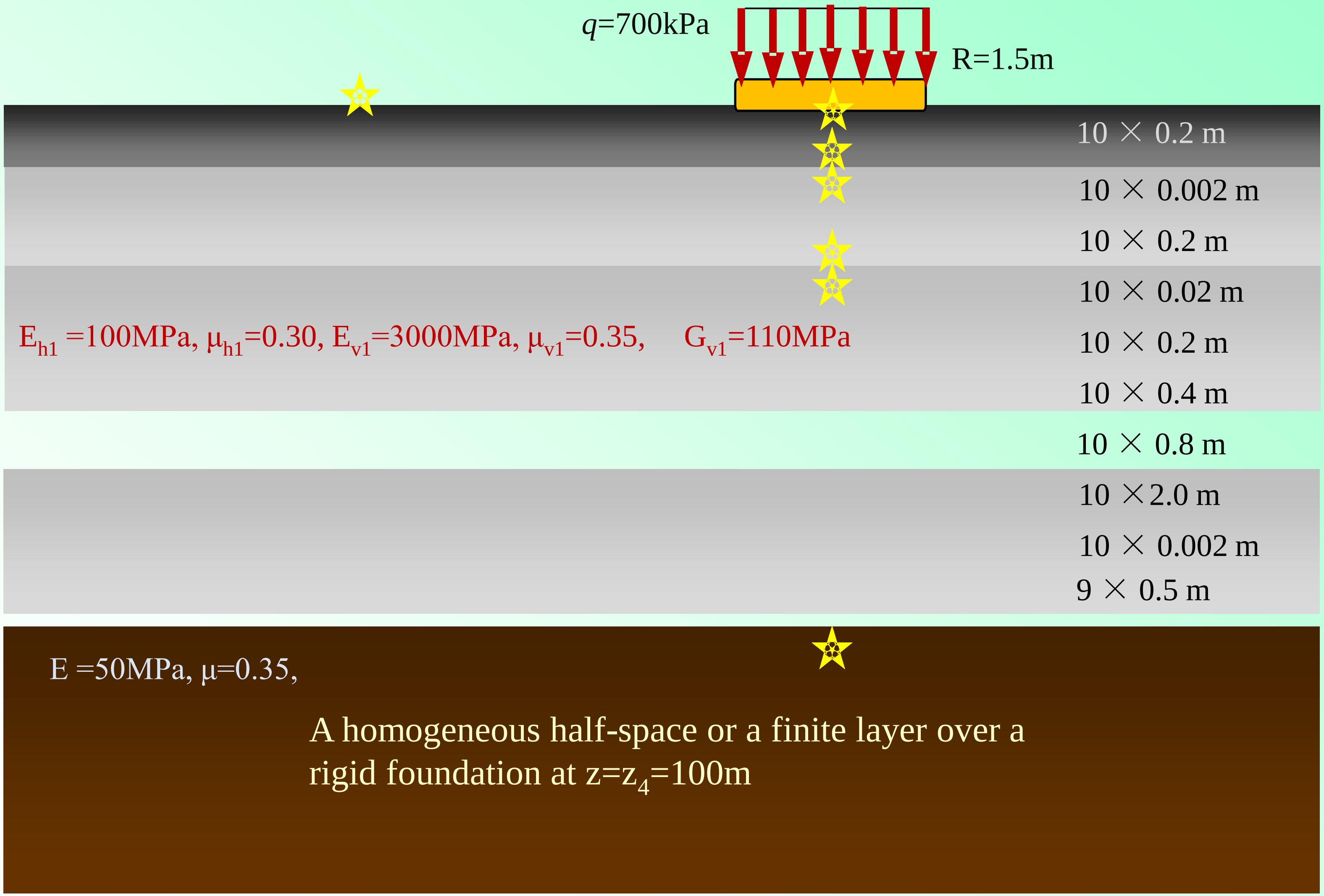
Example 1 (Model 1)



Displacements and stresses at different field points of the layered pavement Model 1 based on the PMM method (u_z in mm/1000 and $\sigma_{\theta\theta}$ in Pa).

Field point (m)			Two layers over a half-space		Three layers over a rigid foundation at $z=z_3=100$m	
x	y	z	u_z	$\sigma_{\theta\theta}$	u_z	$\sigma_{\theta\theta}$
0	0	0	500.5993	-1714959	497.3226	-1714885
0	0.3	0	401.5442	-513353	398.2675	-513303
0	0.6	0	308.2881	-181716	305.0114	-181665
0	0.9	0	242.9199	-86899.8	239.6433	-86849
0	1.2	0	195.1638	-46342.4	191.8873	-46291.6
0	2	0	118.2472	-10699	114.971	-10648.3
0	0	0.1499	491.5508	1232200	488.273	1232250
0	0	0.1501	491.4745	-61081.01	488.1967	-61076.98
0	0	0.4999	358.4996	70188.82	355.2169	70192.79
0	0	0.5001	358.4323	2468.6389	355.1496	2469.2833
0	0	50.5	7.6840	0.3488	4.0186	0.2085
0	0	98.5	3.9902	0.0855	0.0575	-2.0377

Example 3 (Model 3)



Field point (m)		Ninety-nine layers over a half-space					
		PIM		SMM		PMM	
x	z	u_z	$\sigma_{\theta\theta}$	u_z	$\sigma_{\theta\theta}$	u_z	$\sigma_{\theta\theta}$
0	0	8342.734	-311172.268	8342.734	-311172.268	8342.734	-311172.268
0	1.9999	4857.905	-3902.029	4857.905	-3902.029	4857.905	-3902.029
0	2.0199	4830.379	-3130.655	4830.379	-3130.655	4830.379	-3130.655
0	4.2199	2957.549	11418.776	2957.549	11418.776	2957.549	11418.776
0	18.2199	1219.890	938.721	1219.890	938.721	1219.890	938.721
0	38.2199	953.796	921.101	953.796	921.101	953.796	921.101
0	38.2399	953.674	921.957	953.674	921.957	953.674	921.957
0	42.7399	927.724	1150.159	927.724	1150.159	927.724	1150.159
0	50	805.808	101.820	805.808	101.820	805.808	101.820
CPU (s)		**93**		**31**		**6**	

Field point (m)		One hundred layers over a rigid foundation at $z=z_{100}=100$m					
		PIM		SMM		PMM	
x	z	u_z	$\sigma_{\theta\theta}$	u_z	$\sigma_{\theta\theta}$	u_z	$\sigma_{\theta\theta}$
0	0	8061.462	-311096.770	8061.462	-311096.770	8061.462	-311096.770
0	1.9999	4576.288	-3829.914	4576.288	-3829.914	4576.288	-3829.914
0	2.0199	4548.758	-3058.574	4548.758	-3058.574	4548.758	-3058.574
0	4.2199	2675.567	11487.170	2675.567	11487.170	2675.567	11487.170
0	18.2199	935.966	984.581	935.966	984.581	935.966	984.581
0	38.2199	667.890	936.345	667.890	936.345	667.890	936.345
0	38.2399	667.766	937.170	667.766	937.170	667.766	937.170
0	42.7399	641.468	1158.544	641.468	1158.544	641.468	1158.544
0	50	517.232	98.175	517.232	98.175	517.232	98.175
CPU (s)		**103**		**33**		**6**	

Pavement Made of Three Layers (Model 1)

$$G_v = \frac{E_v}{2(1+\mu_v)}$$

Seed modulus Type.	Pavement structure	Actual modulus (MPa)		Seed modulus (MPa)	Inverse modulus (MPa)	Relative error (%)
1	AC	E_{ac}	1500	30000	1500.018	0.001
	Base	E_{bh}	300	3000	299.999	0.000
		E_{bv}	500	3000	499.983	-0.003
	Subgrade	E_{sg}	30	300	30.000	0.000
2	AC	E_{ac}	1500	10000	1500.015	0.001
	Base	E_{bh}	300	10000	300.0	0.000
		E_{bv}	500	10000	499.990	-0.002
	Subgrade	E_{sg}	30	10000	30.000	0.000
3	AC	E_{ac}	1500	300	1500.022	0.001
	Base	E_{bh}	300	3000	299.999	0.000
		E_{bv}	500	3000	499.985	-0.003
	Subgrade	E_{sg}	30	3000	30.000	0.000

Pavement Made of Three Layers (Model 2)

Seed moduli type	Pavement structure	Actual modulus (MPa)		Seed modulus (MPa)	Inverse modulus (MPa)	Relative error (%)
1	AC	E_{ac}	1500	30000	1500.002	0.000
	Base	E_{bh}	300	3000	299.997	-0.001
		E_{bv}	500	5000	499.965	-0.007
		G_v	178.6	3000	178.571	-0.016
	Subgrade	E_{sg}	30	300	30.000	0.000
2	AC	E_{ac}	1500	10000	1500.001	0.000
	Base	E_{bh}	300	10000	299.997	-0.001
		E_{bv}	500	10000	499.972	-0.006
		G_v	178.6	10000	178.571	-0.016
	Subgrade	E_{sg}	30	10000	30.00	0.000
3	AC	E_{ac}	1500	300	1500.002	0.000
	Base	E_{bh}	300	300	299.996	-0.001
		E_{bv}	500	5000	499.964	-0.007
		G_v	178.6	3000	178.571	-0.016
	Subgrade	E_{sg}	30	3000	30.000	0.000

Interface Matrix

$$T_{L-} = T_{L+}$$

$$T_{M-} = T_{M+}$$

$$T_{N-} = T_{N+}$$

$$U_{L-} - U_{L+} = \alpha_z^j T_{L-}$$

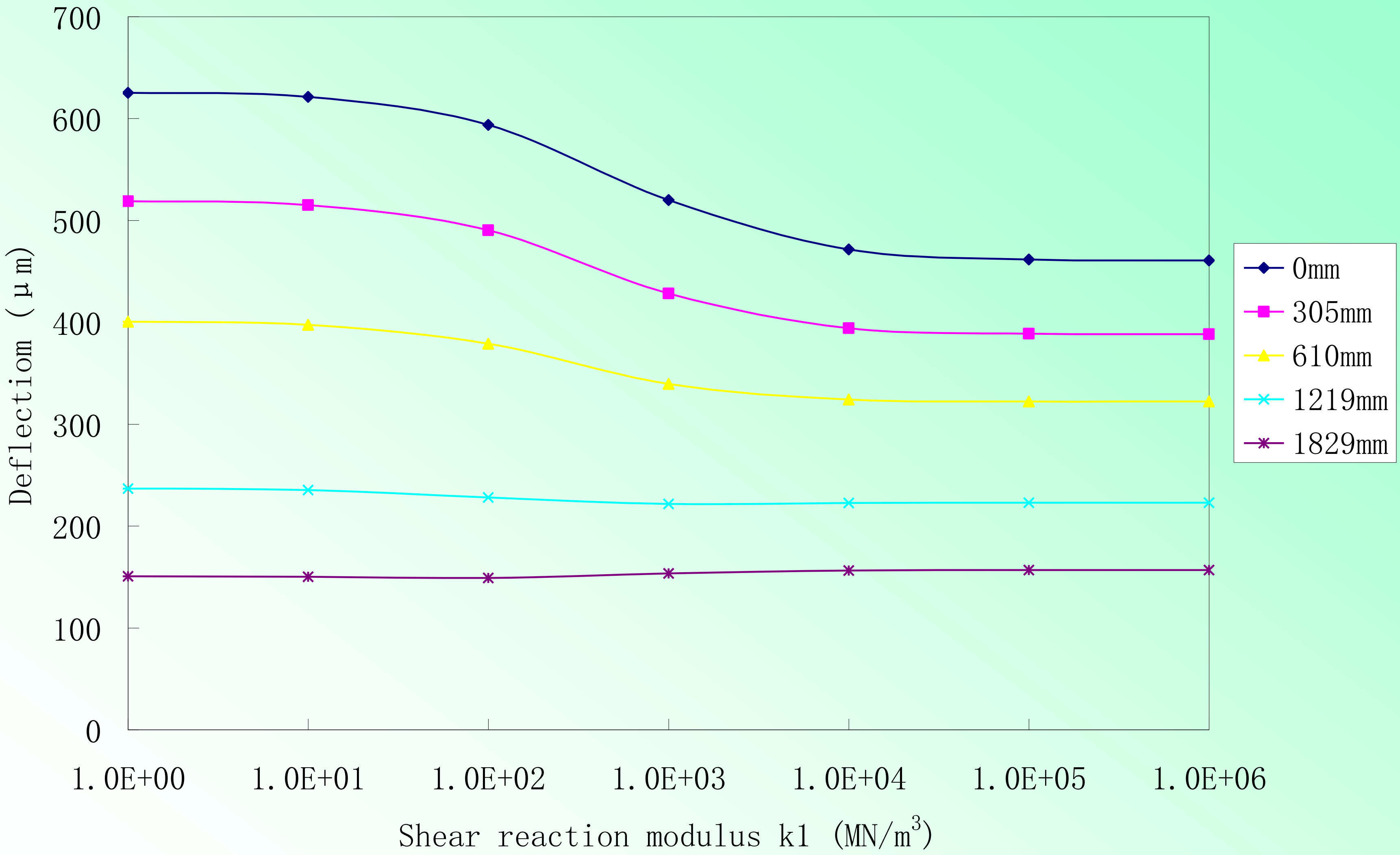
$$U_{M-} - U_{M+} = \alpha_x^j T_{M-}$$

$$U_{N-} - U_{N+} = \alpha_x^j T_{N-}$$

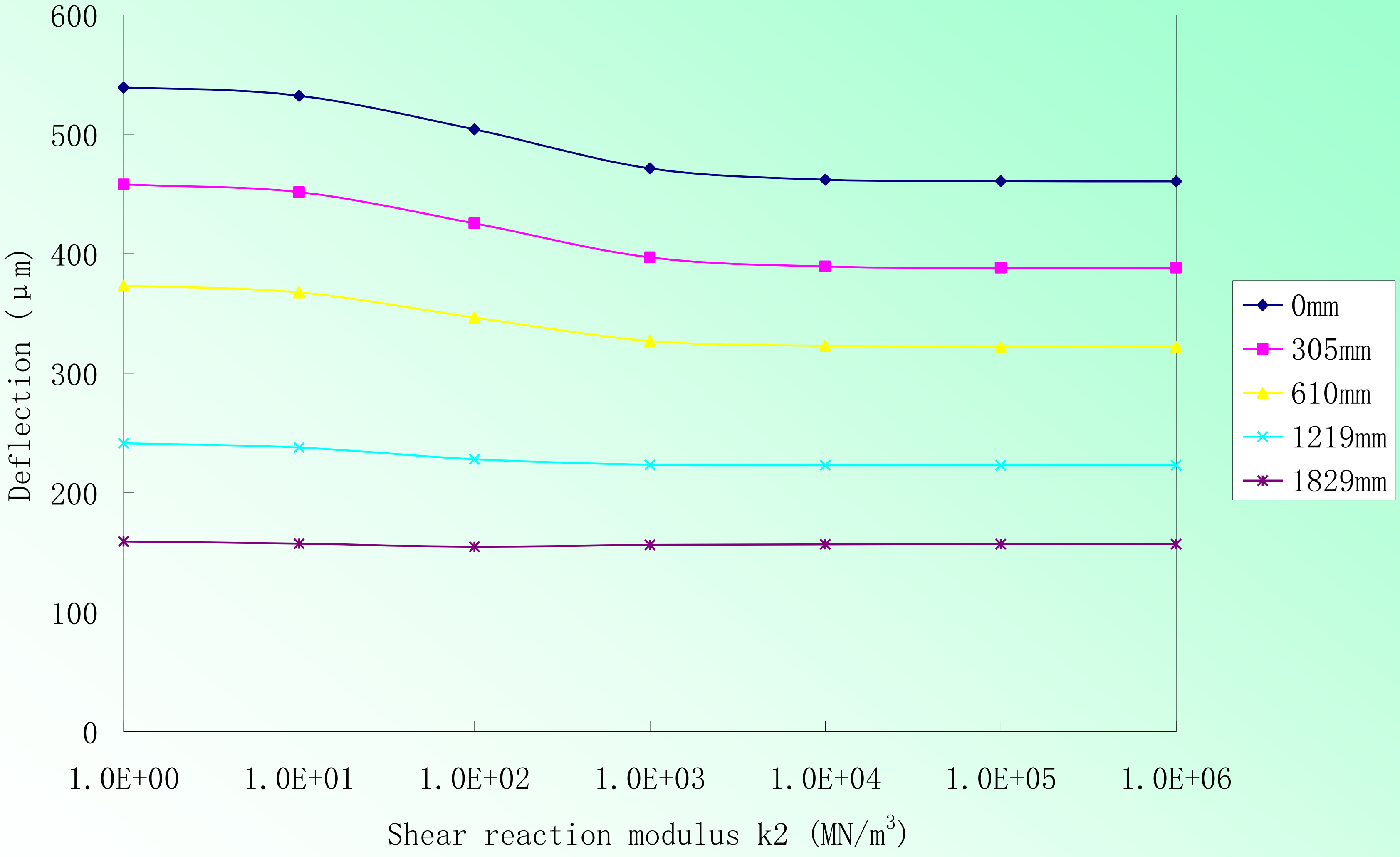
$$\begin{bmatrix} U_L \\ \lambda U_M \\ T_L/\lambda \\ T_M \end{bmatrix}_{z_j-0} = \begin{bmatrix} 1 & 0 & \lambda\alpha_z^{(j)} & 0 \\ 0 & 1 & 0 & \lambda\alpha_x^{(j)} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} U_L \\ \lambda U_M \\ T_L/\lambda \\ T_M \end{bmatrix}_{z_j+0}$$

$$\equiv [InterM] \begin{bmatrix} U_L \\ \lambda U_M \\ T_L/\lambda \\ T_M \end{bmatrix}_{z_j+0}$$

Bonding k1 Between AC & Base



Bonding k2 Between Base & Subbase



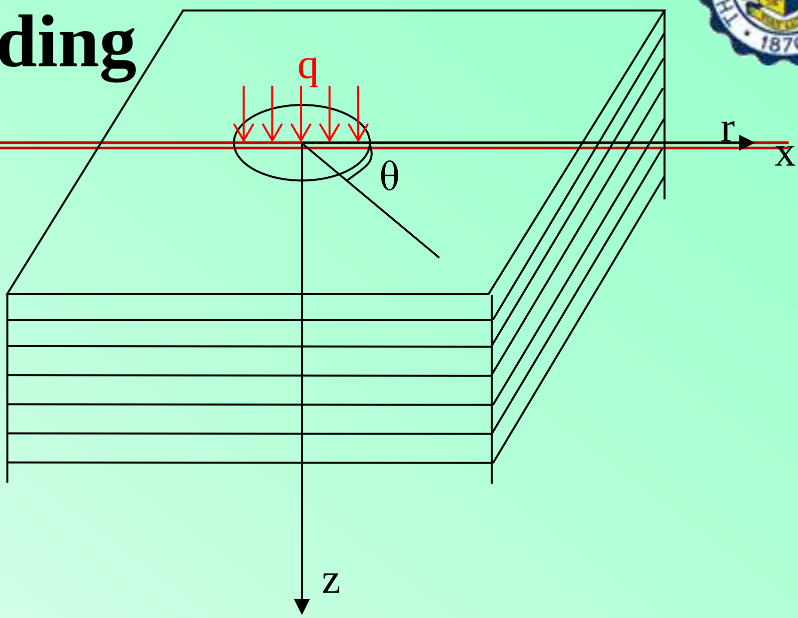
Backcalculation with k1

Pavement	Layer	Seed Value	Real Value	Backcalculated Value	error(%)
1	E_{ac}	30000	5000	4999.988	0.000
	k_1	10000	100	100	0.000
	E_b	3000	1200	1200.013	0.001
	E_{sb}	3000	300	299.998	-0.001
	E_{sg}	300	50	50	0.000
2	E_{ac}	300	5000	4999.969	-0.001
	k_1	10000	100	100	0.000
	E_b	3000	1200	1200.013	0.001
	E_{sb}	3000	300	299.996	-0.001
	E_{sg}	30000	50	50	0.000
3	E_{ac}	10000	5000	5000.069	0.001
	k_1	10000	100	99.997	-0.003
	E_b	10000	1200	1200.009	0.001
	E_{sb}	10000	300	300.002	0.001
	E_{sg}	3000	50	50	0.000

Backcalculation with k2

Pavement	Layer	Seed Value	Real Value	Backcalculated Value	error(%)
1	E_{ac}	10000	5000	5000.01	0.000
	E_b	10000	1200	1199.995	0.000
	k_2	10000	1000	999.668	-0.033
	E_{sb}	10000	300	300.03	0.010
	E_{sg}	1000	50	50	0.000
2	E_{ac}	300	5000	4999.995	0.000
	E_b	3000	1200	1199.989	-0.001
	k_2	100000	1000	1000.008	0.001
	E_{sb}	30000	300	300.009	0.003
	E_{sg}	3000	50	50	0.000
3	E_{ac}	1000	5000	4999.961	-0.001
	E_b	100	1200	1199.982	-0.002
	k_2	10	1000	1000.021	0.002
	E_{sb}	100	300	300.02	0.007
	E_{sg}	10	50	50	0.000

Boundary Conditions of Vertical Axis-Symmetric Circular Loading



$$\sigma_{zz} = \begin{cases} -q & r \le R \\ 0 & r > R \end{cases}$$

$$\sigma_{rz} = \sigma_{\theta z} = 0 \qquad r \ge 0$$

$$\mathbf{P} = \sigma_{rz}(r,\theta,0)\mathbf{e}_r + \sigma_{\theta z}(r,\theta,0)\mathbf{e}_\theta + \sigma_{zz}(r,\theta,0)\mathbf{e}_z = \sigma_{zz}(r,\theta,0)\mathbf{e}_z$$

$$\mathbf{P}\ (r,\theta,z_0) = \sum_m \int_0^\infty P_L(\lambda,m)\mathbf{L}(r,\theta) + P_M(\lambda,m)\mathbf{M}(r,\theta) + P_N(\lambda,m)\mathbf{N}(r,\theta)]\lambda d\lambda$$

$$P_L = -q\sqrt{2\pi}\int_0^R J_0(\lambda r) r dr = -\frac{\sqrt{2\pi} q R J_1(\lambda r)}{\lambda} \qquad P_M = P_N = 0$$

Final Propagation Relation

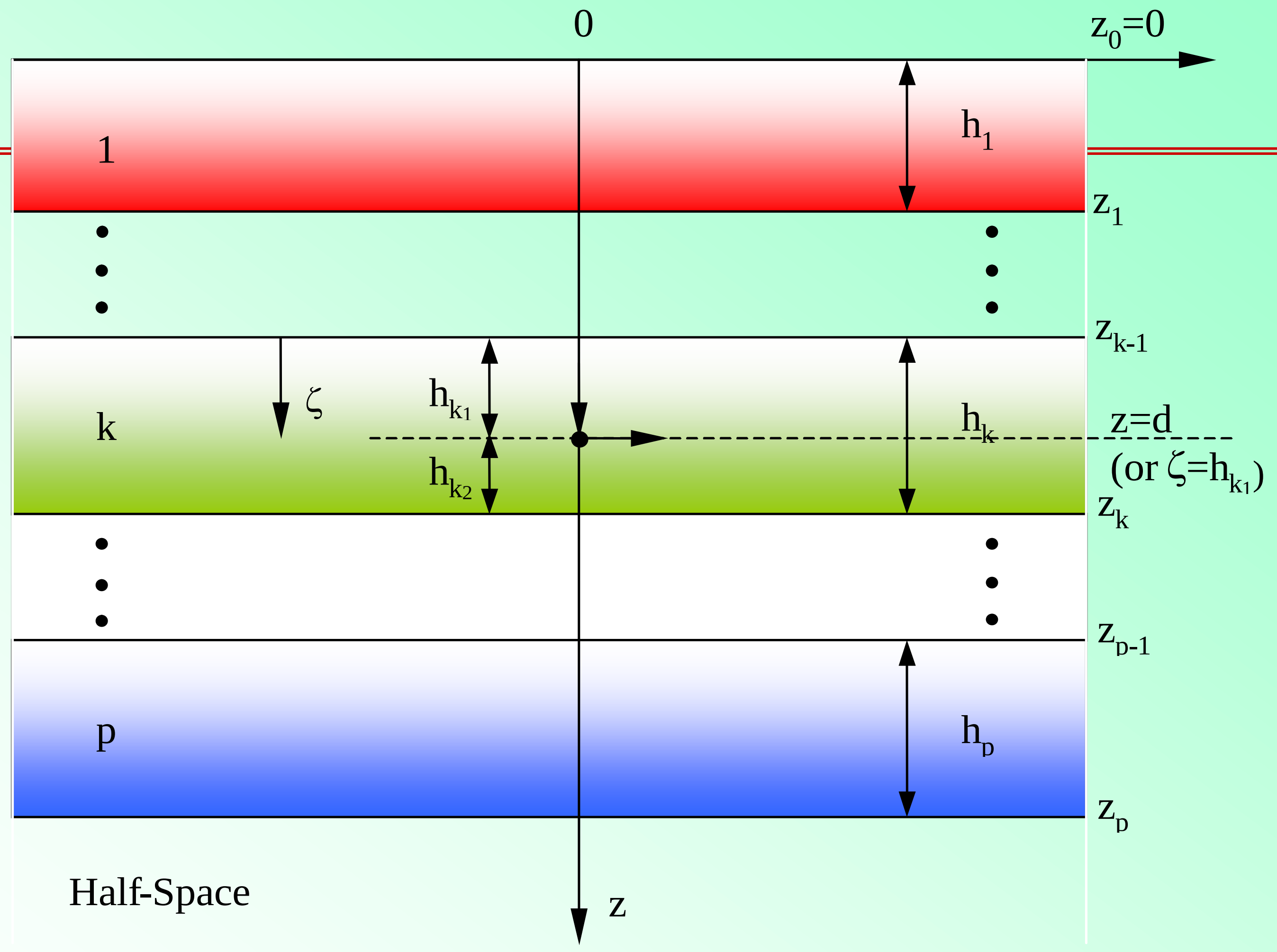
$$[E(z_0)] = \begin{bmatrix} U_L \\ \lambda U_M \\ T_L/\lambda \\ T_M \end{bmatrix}_{z=z_0} = [a_1][a_2]\cdots[a_p][E(z_p)] = [G]\begin{bmatrix} U_L \\ \lambda U_M \\ T_L/\lambda \\ T_M \end{bmatrix}_{z=z_N}$$

$$U_L(z_0) = ?, \qquad T_L(z_0)/\lambda = P_L/\lambda = -\frac{\sqrt{2\pi}qRJ_1(\lambda r)}{\lambda^2}$$

$$\lambda U_M(z_0) = ?, \qquad T_M(z_0) = P_M = 0$$

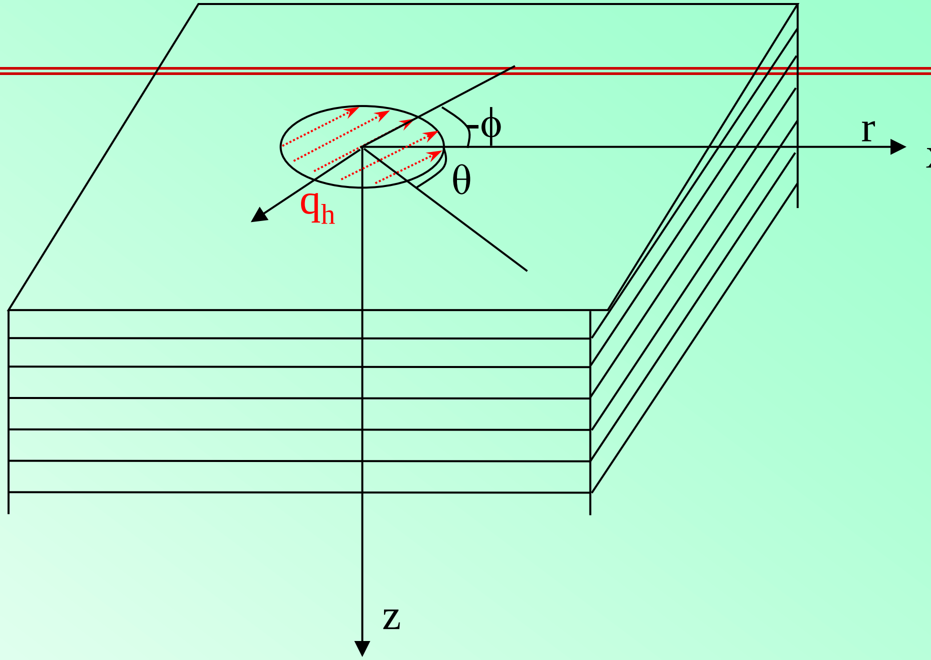
Solving for the unknown displacements on the surface z_0

Solution at any field point z



$$[E(z)] = [a_k(z - z_{k+1})][a_{k+1}]\cdots[a_N][E(z_N)]$$

Boundary Conditions of Horizontal Circular Loading



$$\sigma_{zz} = \begin{cases} 0 & r \le R \\ 0 & r > R \end{cases}$$

$$\sigma_{rz} = \begin{cases} q_h \cdot \cos(\theta - \phi) = q_h(\cos\phi\cos\theta + \sin\phi\sin\theta) & r \le R \\ 0 & r > R \end{cases}$$

$$\sigma_{\theta z} = \begin{cases} -q_h \cdot \sin(\theta - \phi) = -q_h(\sin\varphi\cos\theta - \cos\phi\sin\theta) & r \le R \\ 0 & r > R \end{cases}$$

Expansion Coefficients

$$P_L(\lambda,0) = 0$$

$$P_M(\lambda,0)\Big|_{m=1} = \frac{\sqrt{2\pi}q_h R J_1(R\lambda)}{2\lambda^2}(\cos\phi - i\sin\phi)$$

$$P_M(\lambda,0)\Big|_{m=-1} = \frac{\sqrt{2\pi}q_h R J_1(R\lambda)}{2\lambda^2}(-\cos\phi - i\sin\phi)$$

$$P_N(\lambda,0)\Big|_{m=1} = \frac{\sqrt{2\pi}R q_h}{2\lambda^2} J_1(R\lambda)\cdot(-i\cos\phi - \sin\phi)$$

$$P_N(\lambda,0)\Big|_{m=-1} = \frac{\sqrt{2\pi}R q_h}{2\lambda^2} J_1(R\lambda)\cdot(-i\cos\phi + \sin\phi)$$

Final Propagation Relations

(with both LM- and N- types)

$$U_L(z_0) = ?, \qquad T_L(z_0)/\lambda = P_L/\lambda = 0$$

$$\lambda U_M(z_0) = ?, \qquad T_M(z_0) = P_M$$

$$U_N(z_0) = ? \qquad T_N(z_0)/\lambda = P_N/\lambda$$

$$[E(z_0)] = [a_1][a_2]\cdots[a_{N-1}][Z_{N-1}][0,0,C_N,D_N]^T$$

$$[E^N(z_0)] = [a_1^N][a_2^N]\cdots[a_{N-1}^N][Z_{N-1}][0,B_N^N]^T$$

Expressions For Displacements and Tractions

$$u_r = u_r\big|_{m=1} + u_r\big|_{m=-1} = \frac{2\cos(\theta-\phi)}{\sqrt{2\pi}}\int_0^{+\infty}\{Q_{UM}\cdot[\lambda J_0(\lambda r) - \frac{J_1(\lambda r)}{r}] + \frac{Q_{UN}}{r}J_1(\lambda r)\}\lambda d\lambda$$

$$u_\theta = u_\theta\big|_{m=1} + u_\theta\big|_{m=-1} = \frac{-2\sin(\theta-\phi)}{\sqrt{2\pi}}\int_0^\infty Q_{UM}\frac{J_1(\lambda r)}{r}\lambda d\lambda -$$

$$\frac{2\sin(\theta-\phi)}{\sqrt{2\pi}}\int_0^\infty Q_{UN}(\lambda J_0(\lambda r) - \frac{J_1(\lambda r)}{r})\lambda d\lambda$$

$$u_z = u_z\Big|_{m=1} + u_z\Big|_{m=-1} = \frac{2\cos(\theta-\phi)}{\sqrt{2\pi}}\int_0^\infty Q_{UL}J_1(\lambda r)\lambda d\lambda$$

$$\sigma_{rz} = \sigma_{rz}\big|_{m=1} + \sigma_{rz}\big|_{m=-1} = \frac{2\cos(\theta-\phi)}{\sqrt{2\pi}}\int_0^\infty [Q_{TM}(\lambda J_0(\lambda r) - \frac{J_1(\lambda r)}{r}) + Q_{TN}\frac{J_1(\lambda r)}{r}]\lambda d\lambda$$

$$\sigma_{\theta z} = \sigma_{\theta z}\big|_{m=1} + \sigma_{\theta z}\big|_{m=-1} = \frac{-2\sin(\theta-\phi)}{\sqrt{2\pi}}\int_0^{+\infty}\{Q_{TM}\frac{J_1(\lambda r)}{r} + Q_{TN}[\lambda J_0(\lambda r) - \frac{J_1(\lambda r)}{r}]\}\lambda d\lambda$$

$$\sigma_{zz} = \sigma_{zz}\big|_{m=1} + \sigma_{zz}\big|_{m=-1} = \frac{2\cos(\theta-\phi)}{\sqrt{2\pi}}\int_0^{+\infty} Q_{TL}J_1(\lambda r)\lambda d\lambda$$

Expressions In-plane Stresses

$$\sigma_{rr}=\sigma_{rr}\big|_{m=1}+\sigma_{rr}\Big|_{m=-1}=\frac{C_{13}}{C_{33}}\sigma_{zz}-\frac{C_{11}-C_{12}}{\sqrt{2\pi}}\cdot 2\cdot\cos(\theta-\phi)\int_0^\infty Q_{UN}(\frac{\lambda}{r}J_2(\lambda r))\lambda d\lambda$$

$$+\frac{C_{11}-C_{12}}{\sqrt{2\pi}}2\cdot\cos(\theta-\phi)\int_0^\infty Q_{UM}(\frac{\lambda}{r}J_2(\lambda r))\lambda d\lambda$$

$$+\frac{C_{13}^2-C_{11}C_{33}}{C_{33}\sqrt{2\pi}}2\cdot\cos(\theta-\phi)\int_0^\infty Q_{UM}J_1(\lambda r)\lambda^3 d\lambda$$

$$\sigma_{\theta\theta}=\sigma_{\theta\theta}\big|_{m=1}+\sigma_{\theta\theta}\big|_{m=-1}=\frac{C_{13}}{C_{33}}\sigma_{zz}+\frac{C_{13}^2-C_{12}C_{33}}{C_{33}\sqrt{2\pi}}2\cos(\theta-\phi)\int_0^\infty Q_{UM}J_1(\lambda r)\lambda^3 d\lambda$$

$$+\frac{C_{11}-C_{12}}{\sqrt{2\pi}}2\cos(\theta-\phi)\int_0^\infty Q_{UN}(\frac{\lambda J_2(\lambda r)}{r})\lambda d\lambda$$

$$-\frac{C_{11}-C_{12}}{\sqrt{2\pi}}2\cos(\theta-\phi)\int_0^\infty Q_{UM}(\frac{\lambda J_2(\lambda r)}{r})\lambda d\lambda$$

$$\sigma_{r\theta}=\sigma_{r\theta}\big|_{m=1}+\sigma_{r\theta}\big|_{m=-1}=\frac{4C_{66}}{\sqrt{2\pi}}\sin(\theta-\phi)\int_0^\infty Q_{UM}(\frac{\lambda}{r}J_2(\lambda r))\lambda d\lambda$$

$$+\frac{2C_{66}\sin(\theta-\phi)}{\sqrt{2\pi}}\int_0^\infty Q_{UN}J_1(\lambda r)\lambda^3 d\lambda-\frac{4C_{66}\sin(\theta-\phi)}{\sqrt{2\pi}}\int_0^\infty Q_{UN}(\frac{\lambda J_2(\lambda r)}{r})\lambda d\lambda$$

Multilayered System

$$\begin{bmatrix} \tilde{\mathbf{u}}(z_{j-1}) \\ \tilde{\mathbf{t}}(z_{j-1}) \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} & \bar{\boldsymbol{A}} < \mathrm{e}^{-\bar{s}^{*}\lambda(z_{j-1}-z_j)} > \\ \boldsymbol{B} & \bar{\boldsymbol{B}} < \mathrm{e}^{-\bar{s}^{*}\lambda(z_{j-1}-z_j)} > \end{bmatrix} \begin{bmatrix} \boldsymbol{K}^{+} \\ \boldsymbol{K}^{-} \end{bmatrix}$$

$$\begin{bmatrix} \tilde{\mathbf{u}}(z_{j}) \\ \tilde{\mathbf{t}}(z_{j}) \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} < \mathrm{e}^{-s^{*}\lambda(z_j-z_{j-1})} > & \bar{\boldsymbol{A}} \\ \boldsymbol{B} < \mathrm{e}^{-s^{*}\lambda(z_j-z_{j-1})} > & \bar{\boldsymbol{B}} \end{bmatrix} \begin{bmatrix} \boldsymbol{K}^{+} \\ \boldsymbol{K}^{-} \end{bmatrix}$$

Or

$$\begin{bmatrix} \tilde{\mathbf{u}}(z_{j-1}) \\ \tilde{\mathbf{u}}(z_{j}) \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} & \bar{\boldsymbol{A}} < \mathrm{e}^{-\bar{s}^{*}\lambda(z_{j-1}-z_j)} > \\ \boldsymbol{A} < \mathrm{e}^{-s^{*}\lambda(z_j-z_{j-1})} > & \bar{\boldsymbol{A}} \end{bmatrix} \begin{bmatrix} \boldsymbol{K}^{+} \\ \boldsymbol{K}^{-} \end{bmatrix}$$

$$\begin{bmatrix} \tilde{\mathbf{t}}(z_{j-1}) \\ \tilde{\mathbf{t}}(z_{j}) \end{bmatrix} = \begin{bmatrix} \boldsymbol{B} & \bar{\boldsymbol{B}} < \mathrm{e}^{-\bar{s}^{*}\lambda(z_{j-1}-z_j)} > \\ \boldsymbol{B} < \mathrm{e}^{-s^{*}\lambda(z_j-z_{j-1})} > & \bar{\boldsymbol{B}} \end{bmatrix} \begin{bmatrix} \boldsymbol{K}^{+} \\ \boldsymbol{K}^{-} \end{bmatrix}$$

Final Stiffness Matrix & BCs

$$\begin{bmatrix} \boldsymbol{T}(z_0) \\ \boldsymbol{T}(z_N) \end{bmatrix} = [\boldsymbol{K}_{1:N}] \begin{bmatrix} \boldsymbol{U}(z_0) \\ \boldsymbol{U}(z_N) \end{bmatrix}$$

Boundary conditions on the surface

$$\boldsymbol{T}(z_0) = [T_L(z_0)/\lambda, T_M]$$
$$\boldsymbol{U}(z_0) = ?$$

Boundary conditions on the bottom

$$\begin{bmatrix} \boldsymbol{U}(z_N) \\ \boldsymbol{T}(z_N) \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}_2 \boldsymbol{K}^- \\ \boldsymbol{B}_2 \boldsymbol{K}^- \end{bmatrix} \quad \text{vs.} \quad \boldsymbol{U}(z_N)=\boldsymbol{0}$$

Half-space vs. rigid base

Final Solutions

For half-space

$$U(z_n) = [R_\infty + K_{21}^{1:n}(K_{11}^{1:n})^{-1}K_{12}^{1:n} - K_{22}^{1:n}]^{-1}[K_{21}^{1:n}][K_{11}^{1:n}]^{-1}T(z_0)$$

$$U(z_0) = [K_{11}^{1:n}]^{-1}\left\{T(z_0) - [K_{12}^{1:n}]U(z_n)\right\}$$

$$T(z_n) = [R_\infty]U(z_n) \quad [R_\infty] = [W_{22}][W_{12}]^{-1}$$

For rigid foundation

$$U(z_0) = [K_{11}^{1:n+1}]^{-1}T(z_0)$$

$$T(z_{n+1}) = [K_{21}^{1:n+1}]U(z_0) = [K_{21}^{1:n+1}][K_{11}^{1:n+1}]^{-1}T(z_0)$$

Expansion Coefficients

---approximation

$$g_1 = -D; \quad 2g_2 = Ag_1 - g_1C; \quad 3g_3 = Ag_2 - g_2C + g_1Bg_1$$

$$4g_4 = Ag_3 - g_3C + g_1Bg_2 + g_2Bg_1$$

$$f_1 = A; \quad 2f_2 = Af_1 + g_1B; \quad 3f_3 = Af_2 + g_2B + g_1Bf_1$$

$$4f_4 = Af_3 + g_3B + g_2Bf_1 + g_1Bf_2$$

$$e_1 = -C; \quad 2e_2 = Bg_1 - e_1C; \quad 3e_3 = e_1Bg_1 + Bg_2 - e_2C$$

$$4e_4 = e_2Bg_1 + e_1Bg_2 + Bg_3 - e_3C$$

$$q_1 = -B; \quad 2q_2 = -(e_1B + Bf_1); \quad 3q_3 = -(e_2B + e_1Bf_1 + Bf_2)$$

$$4q_4 = -(e_3B + e_1Bf_2 + e_2Bf_1 + Bf_3)$$

Simplification For Conservative Systems

$$C = -A^t; \quad D = D^t; \quad B = B^t$$

$$G = G^t; \quad Q = Q^t; \quad F = E^t$$

State vectors can be simplified as

$$U_b = FU_a - GT_b$$

$$T_a = QU_a + F^t T_b$$

Four Layers (Model 2)

NO.	Layer	Parameter	Seed modulus (MPa)	Inverse modulus (MPa)	Error (%)
1	AC	E_{ach}	20000	1499.516	-0.032
		E_{acv}	30000	2000.143	0.007
	Base	E_{bh}	3000	502.524	0.505
		E_{bv}	4000	799.634	-0.046
	Subbase	E_{sbh}	3000	299.676	-0.108
		E_{sbv}	4000	500.497	0.099
	Subgrade	E_{sg}	300	30	0.000
2	AC	E_{ach}	10000	1498.959	-0.069
		E_{acv}	10000	2000.299	0.015
	Base	E_{bh}	10000	505.816	1.163
		E_{bv}	12000	799.156	-0.106
	Subbase	E_{sbh}	10000	299.253	-0.249
		E_{sbv}	12000	501.174	0.235
	Subgrade	E_{sg}	10000	30	0.000
3	AC	E_{ach}	300	1499.483	-0.034
		E_{acv}	330	2000.151	0.008
	Base	E_{bh}	3000	502.425	0.485
		E_{bv}	3300	799.626	-0.047
	Subbase	E_{sbh}	3000	299.669	-0.110
		E_{sbv}	3300	500.51	0.102
	Subgrade	E_{sg}	30000	30	0.000

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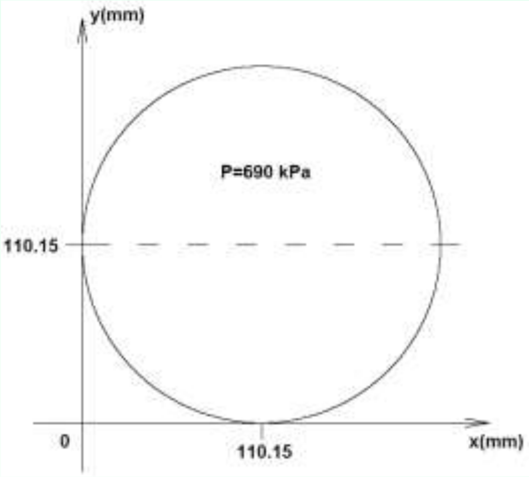
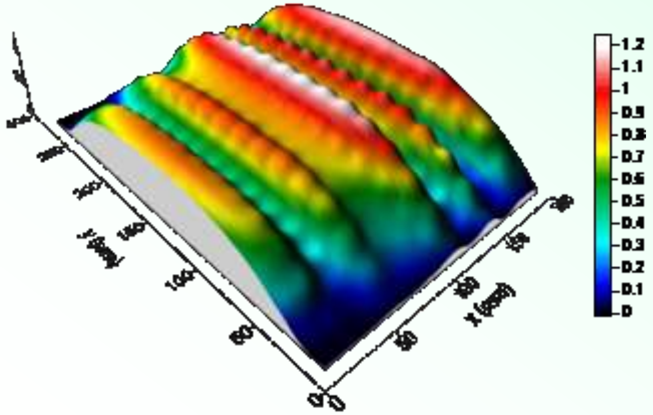
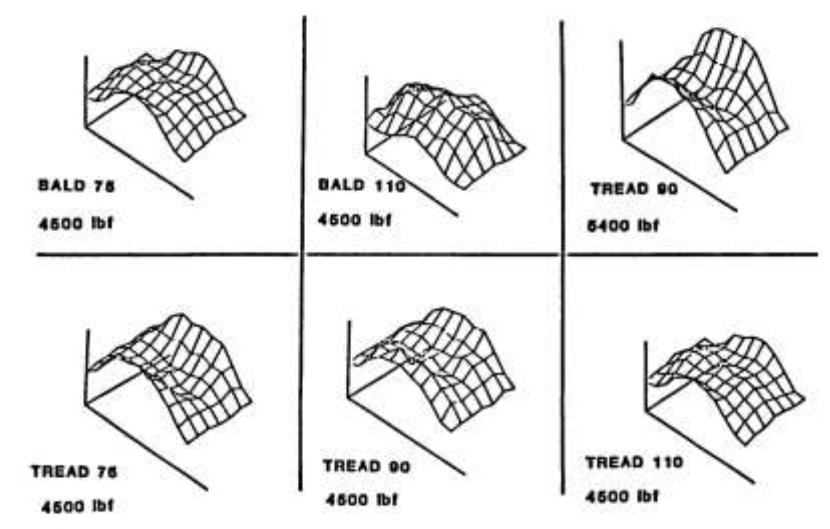
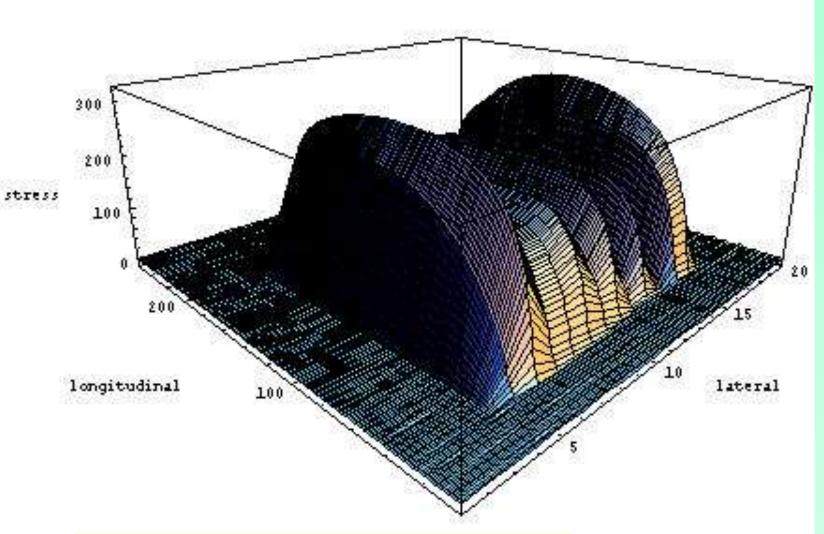
Actual Tire Contact Pressure

Circular vs. Actual Pressure Models





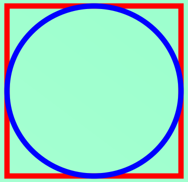


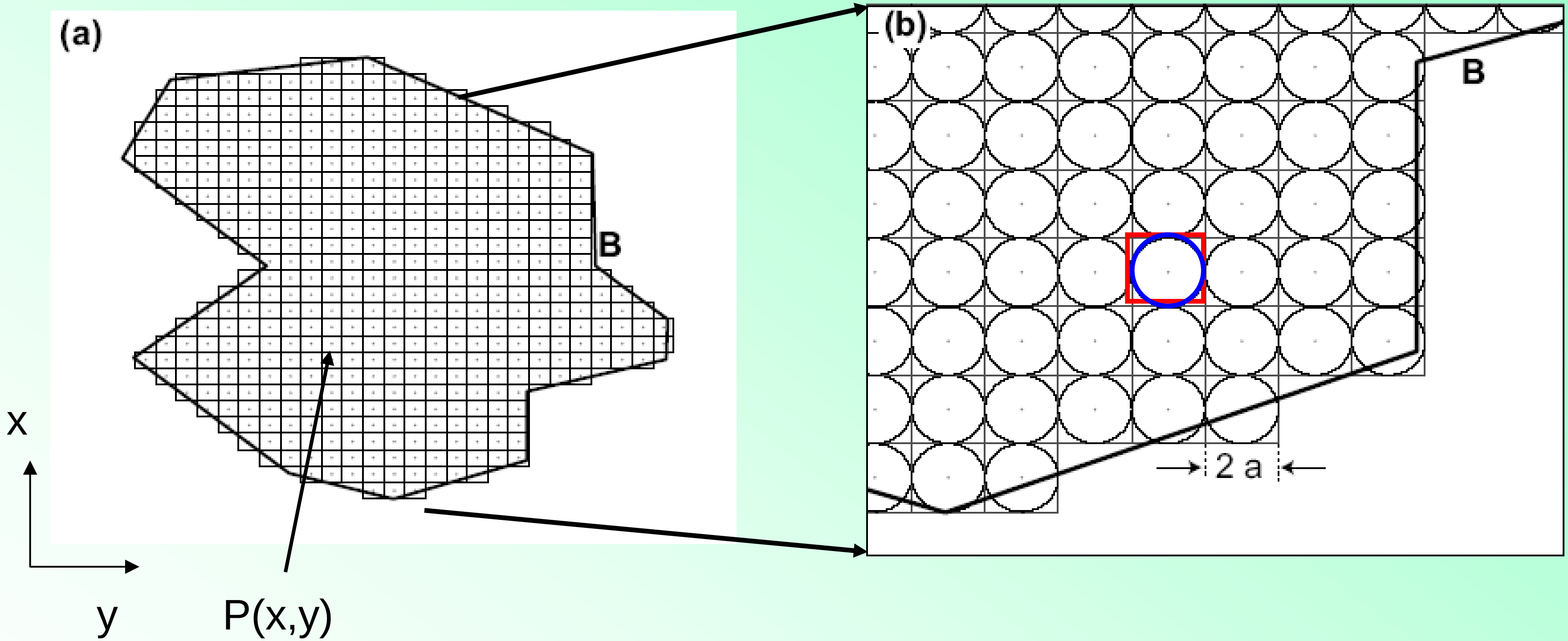


Other Factors

A Superfast Approach…

$$v^j = \sum_{i=1,n} Q_i \, T^{(i)} d^{(i)} = \frac{4}{\pi} \sum_{i=1,n} p_i \, T^{(i)} d^{(i)}$$





$d^{(i)}$: the solution for unit uniform circular loading

$T^{(i)}$: transformation matrix

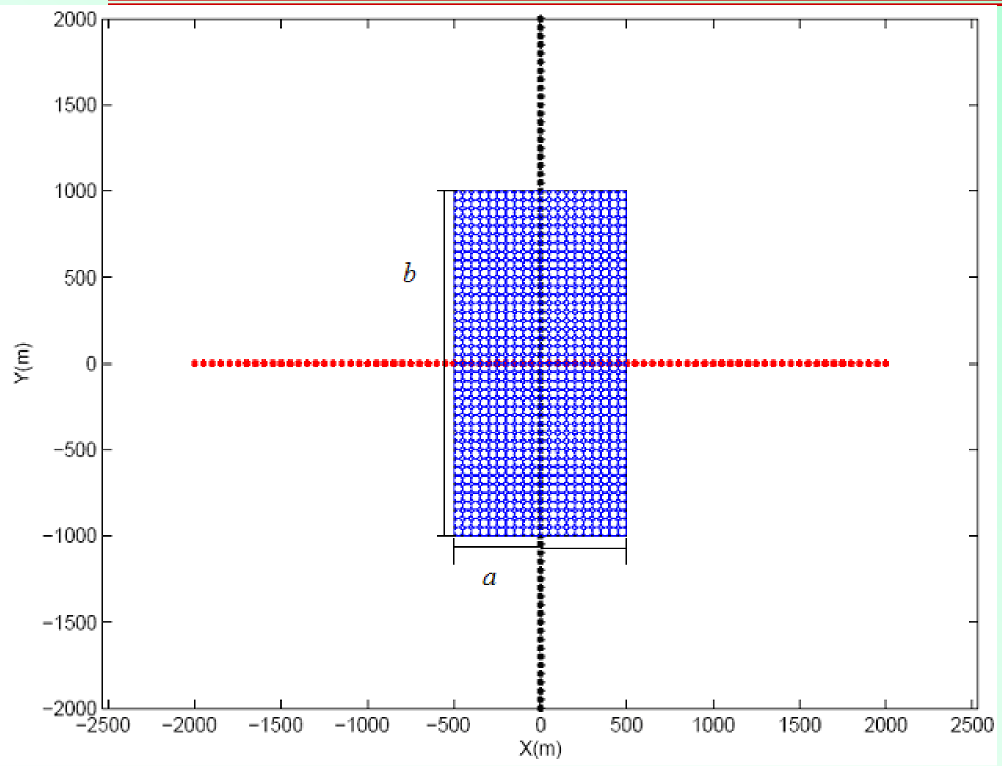
Q_i: uniform pressure inside circle

p_i: uniform pressure inside square

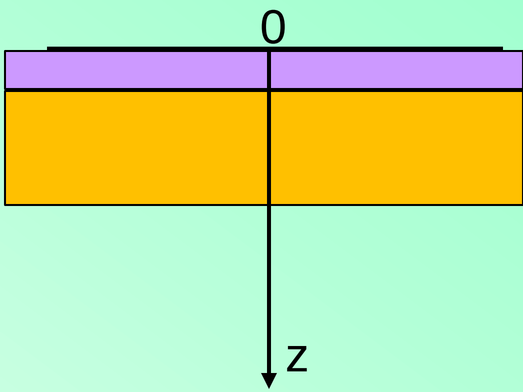
v: total response

Pan et al. (J. Geophys. Int., 2007)

An Example …

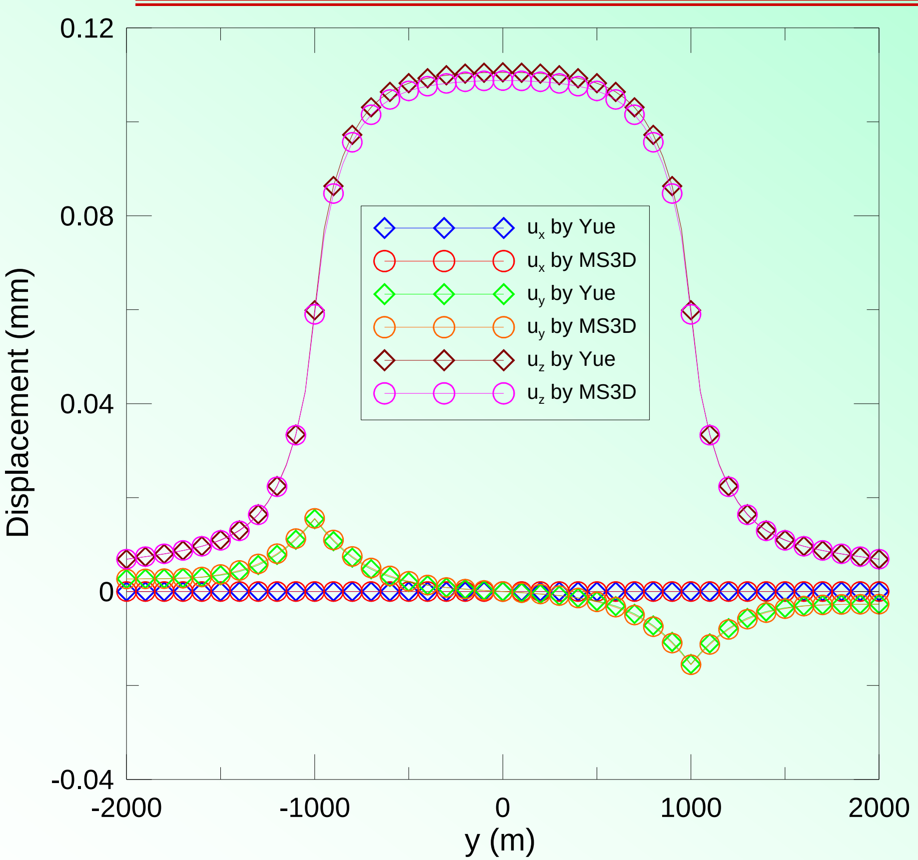


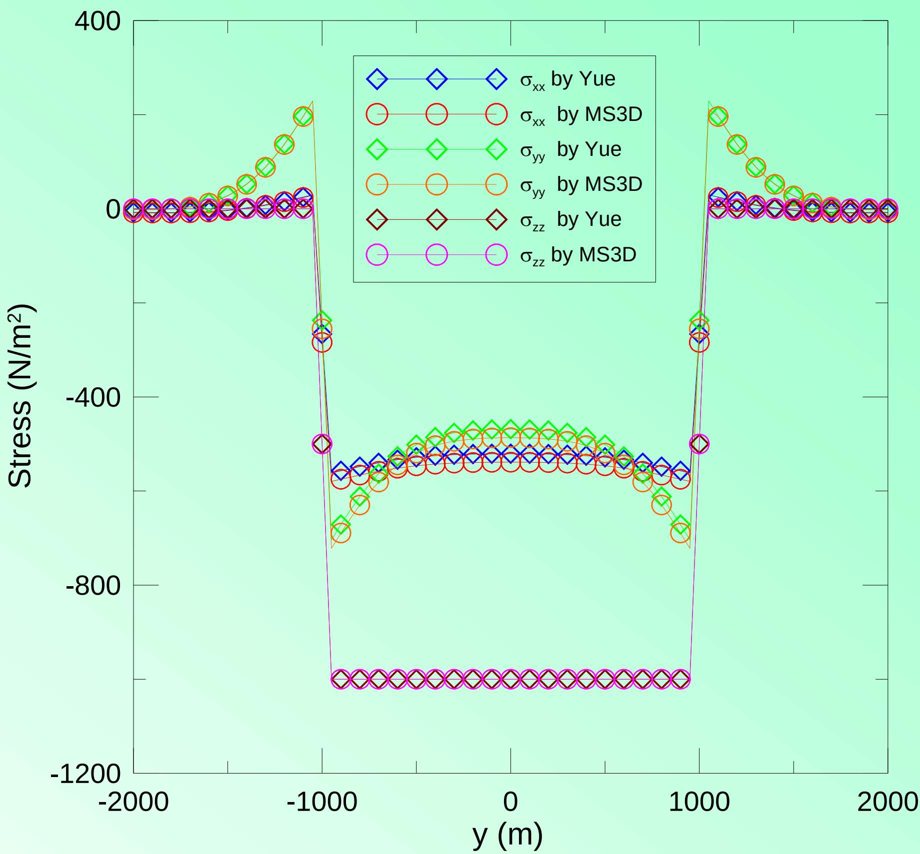
20×40 circular loading cells



Layer #	Thickness (km)	Young's modulus E (GPa)	Poisson's Ratio ν
1	0.5	5.0	0.3
2	5	30.0	0.25
3	infinite	150.0	0.2

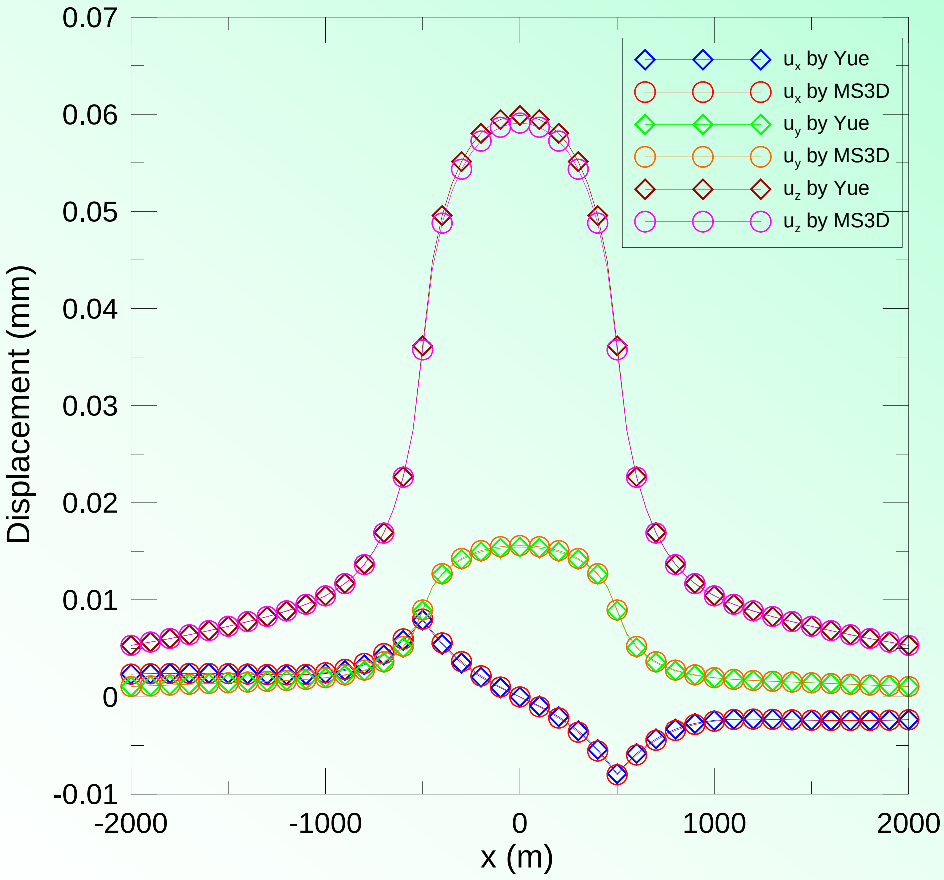
Displacements and Stresses

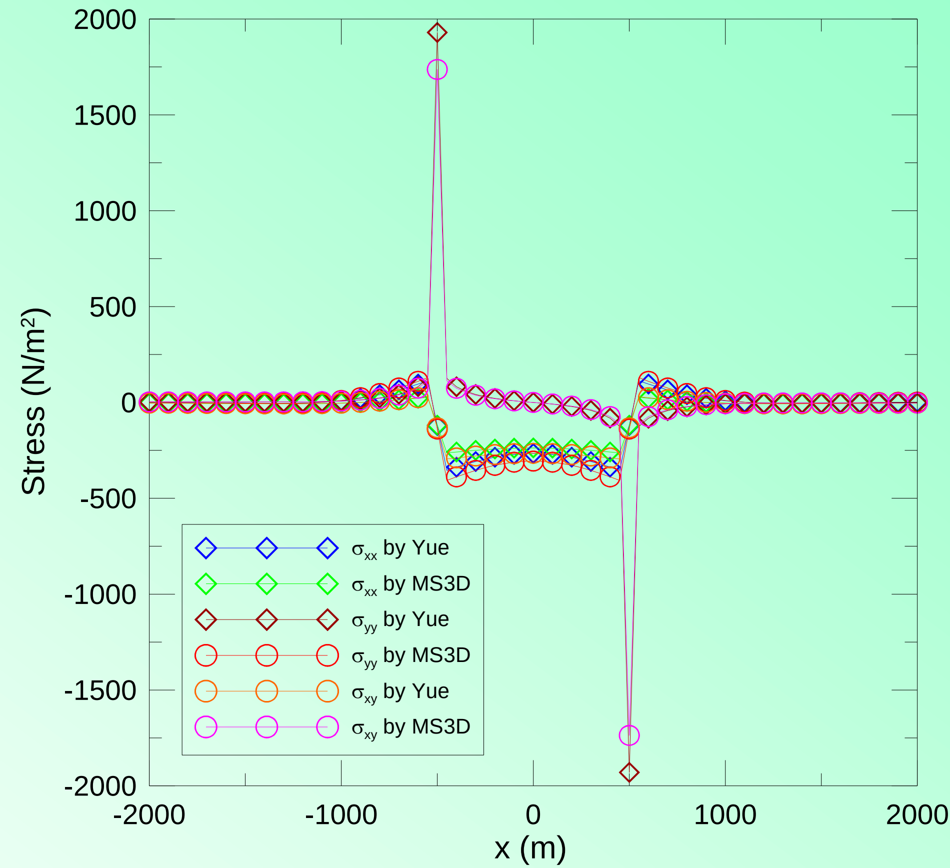




Variation along x-axis

Displacements and Stresses





Variation along y-axis

Conclusions

Special Features of MultiSmart3D

Layered pavement (Green, ODOT); Rock mechanics (Liao, Wang, NCTU); Layered Earth (Bevis, OSU)

1). ***Propagator matrix method*** is introduced so that one needs only to solve two 2×2 systems of linearly algebraic equations in the transformed domain, no matter how many layers we have in the layered structure!
2). ***Cylindrical system of vector functions*** is introduced so that the axisymmetric deformation can be exactly separated from the other part of the deformation!
3). ***Adaptive Gauss quadrature*** is utilized along with an acceleration approach for fast and accurate calculation of the integration!
4). ***Arbitrary observation point*** that can be at any location, far or near; immediately below or above any interface!
5). ***Arbitrary interface*** that can be assumed to have any different shear spring constant!
6). ***Arbitrary surface loading/geometry with super-fast calculation!***
7). The research group has more than **20 years experience on the layered structure modeling and published more than 100 peer-reviewed journal papers in this specific area!**

US Patent 7627428 (12/01/09); Pan et al. (2007, GJI88, 90); …

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--- General Formulation and Solutions

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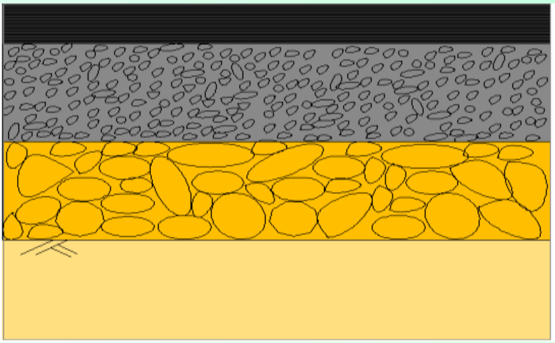
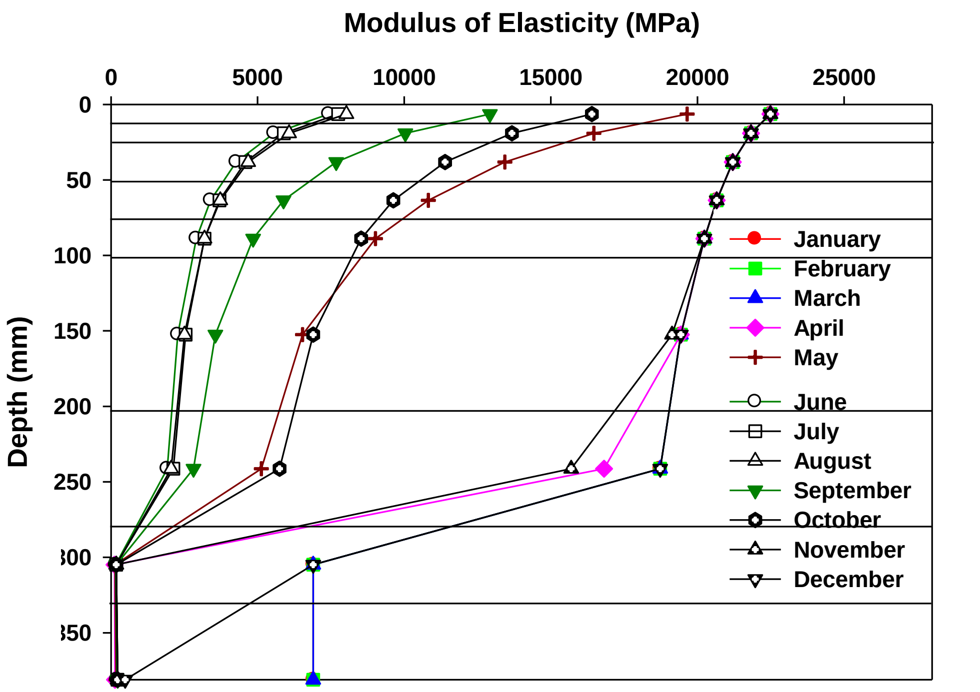
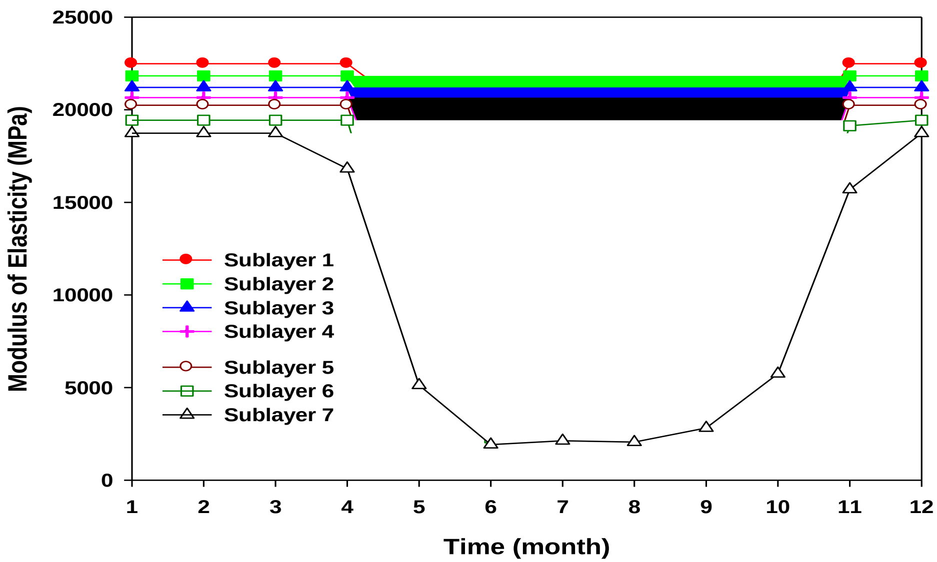
Modulus Variation in AC Layer





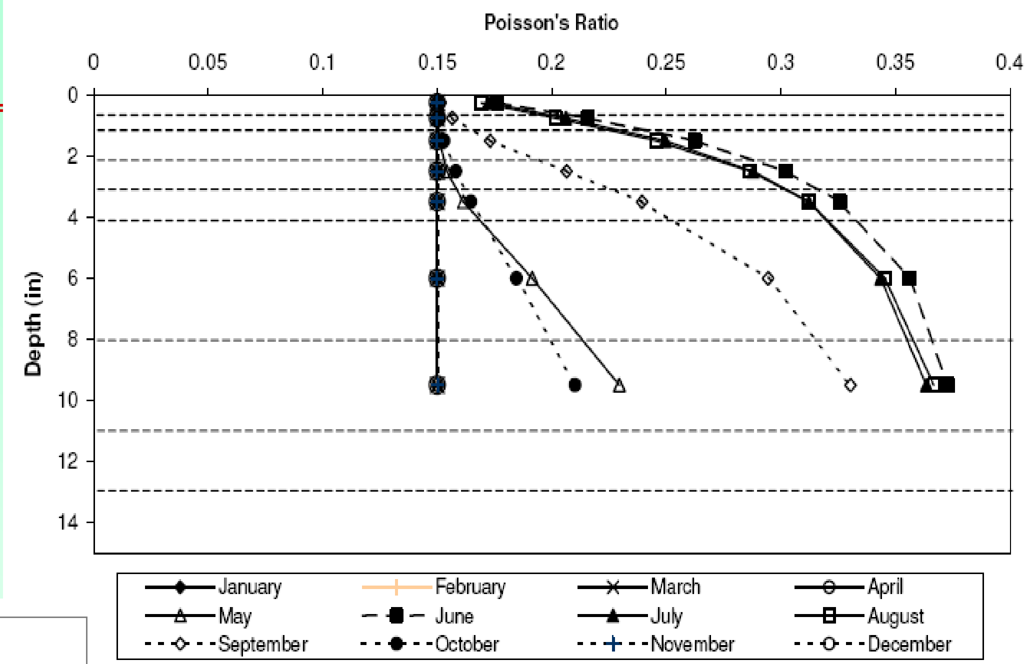
Variation with Depth



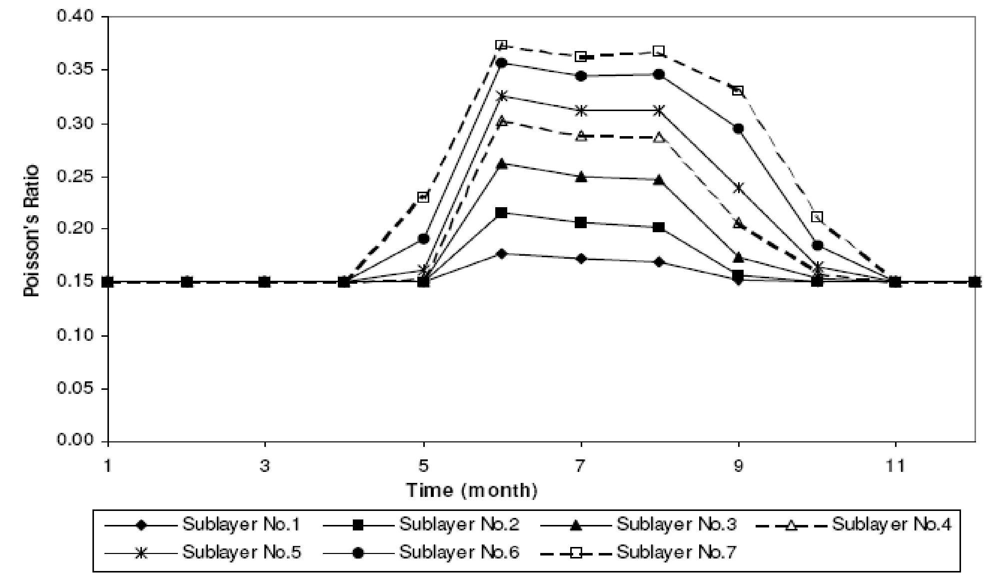
Variation with Time

Using 2004 MEPDG (IOWA)

Seasonal Variation of Poisson's Ratio



Variation with Depth



Variation with Time

MEPDG (2004) Recommendation

Pavement Temperature Variation

Constant values

Barker et al. (1977)

$$T_{asphalt} = 1.2 * T_{air} + 3.2$$

temperature in ° C

$$T_{asphalt} = 1.19 * T_{air}$$

temperature in ° F

Edwards & Valkering (1974) and Ullidtz (1987)

$$T = \frac{(T_1 + T_2)}{2} + \frac{(T_1 - T_2)}{2} \cos\left(\frac{(U - U_O)}{26} * \pi \right)$$

° C or ° F

George and Husain (1986)

$$T_{asp} = T_{air}\left(1 + \frac{76.2}{h_{asp} + 304.8}\right) - \frac{84.7}{h_{asp} + 304.8} + 3.3$$

temperature in ° C

Temperature vs. Time

Lukanen et al. (2000)

temperature in ° C

BELLS equation:

$$T_d = 2.8 + 0.894 * IR + \{\log_{10}(d) - 1.5\}\{-0.54 * IR + 0.770 * (5 - day) + 3.763 * \sin(hr - 18)\}$$
$$+ \{\sin(hr - 14)\}\{0.474 + 0.031 * IR\}$$

BELLS2 equation: **(used widely for flexible pavements)**

$$T_d = 2.78 + 0.912 * IR + \{\log_{10}(d) - 1.25\}\{-0.428 * IR + 0.553 * (1 - day) + 2.63 * \sin(hr_{18} - 15.5)\}$$
$$+ 0.027 * IR * \sin(hr_{18} - 13.5)$$

BELLS3 equation:

$$T_d = 0.95 + 0.892 * IR + \{\log_{10}(d) - 1.25\}\{-0.448 * IR + 0.621 * (1 - day) + 1.83 * \sin(hr_{18} - 15.5)\}$$
$$+ 0.042 * IR * \sin(hr_{18} - 13.5)$$

Pavement Temperature Variation

Temperature gradient

Ongel & Harvey (2004) temperature in ° C ⇐**Based on EICM**

for pavement surface to quarter-depth **thermal gradient**:

$$TQ = -41.7 + 2.08 * T - 1.47 * t + 19.5\left[\sin(hr - 10) * 2 * \frac{\pi}{24}\right]$$

for pavement quarter-depth to mid-depth **thermal gradient**:

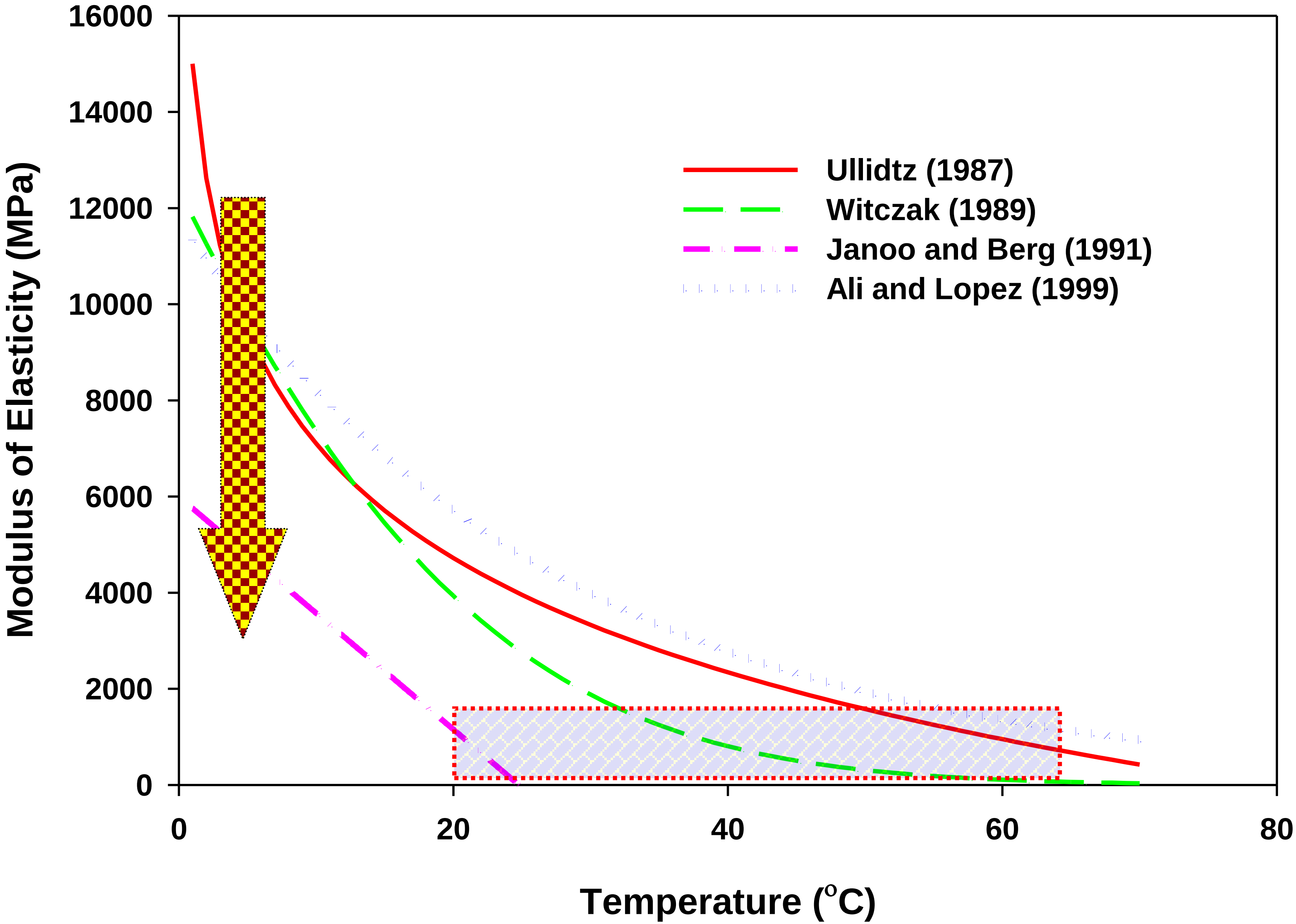
$$TQ = -46.1 + 2.278 * T + 67 * t + 16.18\left[\sin(hr - 10) * 2 * \frac{\pi}{24}\right] - 3.146 * T * t$$

Modulus vs. Temperature

Ullidtz (1987)	$E = 15{,}000 - 7900\log(t)$	MPa and ° C
Witczak (1989)	$E = 10^{(6.53658 - 0.006447*T - 0.00007404*T^2)}$	Psi and ° F
Janoo and Berg (1991)	$E = 5994 - 242*T$	MPa and ° C
Ali and Lopez (1999)	$E = e^{(9.37196 - 0.03608145*T)}$	MPa and ° C

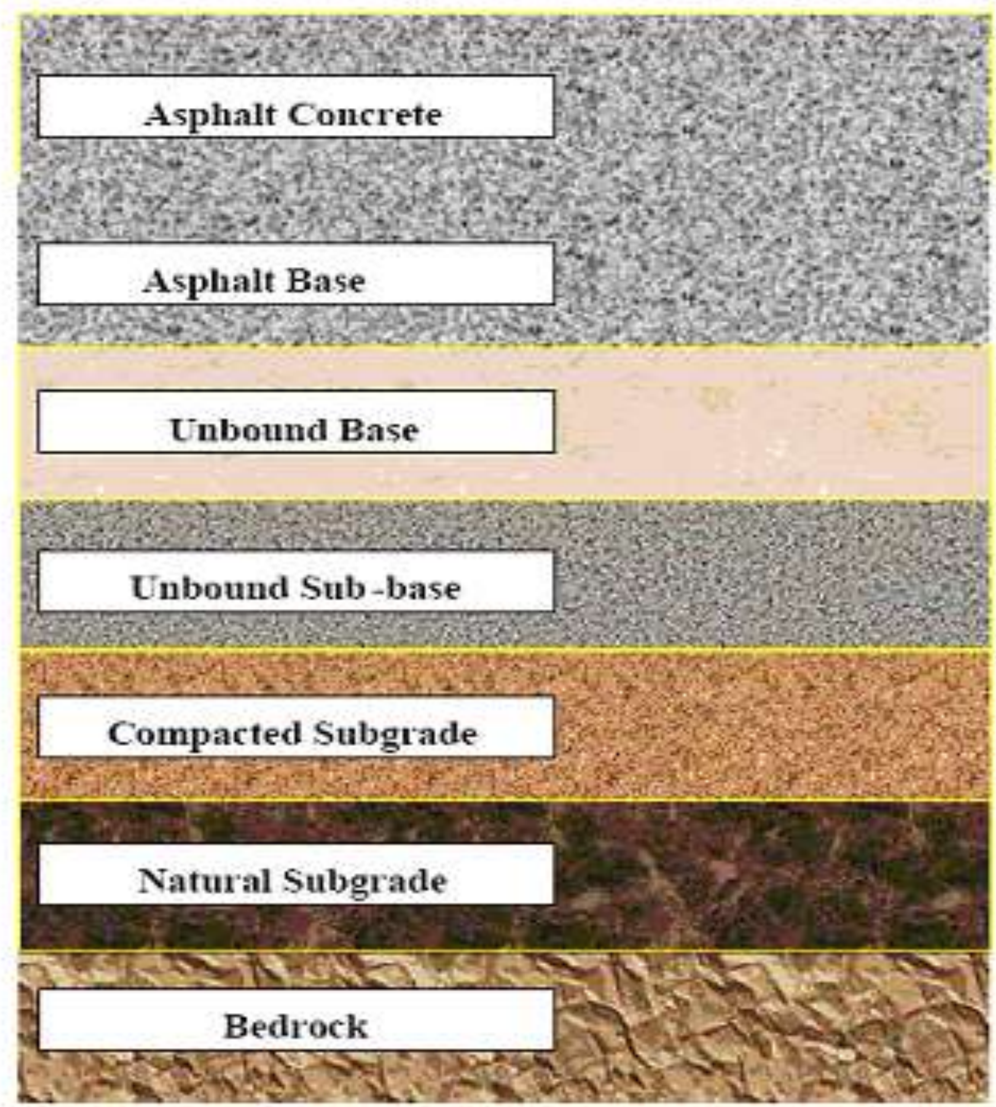
For AC based on lab & field data

Modulus vs. Temperature



Pavement Sublayering in MEPDG

Maximum Sublayering = 2.44 m



12.7 mm

12.7 mm

25.4 mm increments to depth of 102 mm

Remaining thickness

$51\,\text{mm}\ \{\text{if } h_B \geq 152\,\text{mm}\}$

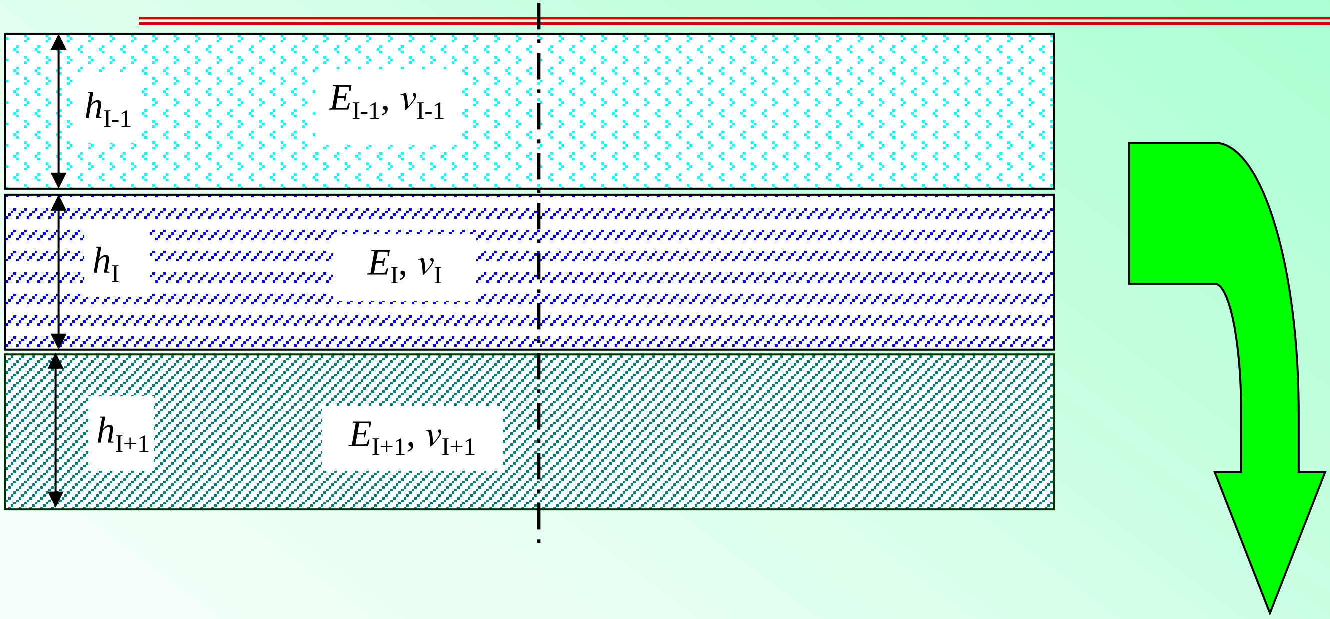
$$n_B = \text{int}\left(\frac{h_B - 51}{4}\right) \text{for } h_B > 152\,\text{mm}$$

$$n_{SB} = \text{int}\left(\frac{h_{SB}}{4}\right) \text{for } h_{SB} \geq 203\,\text{mm}$$

$$n_{CSG} = \text{int}\left(\frac{h_{CSG}}{12}\right) \text{for } h_{CSG} > 305\,\text{mm}$$

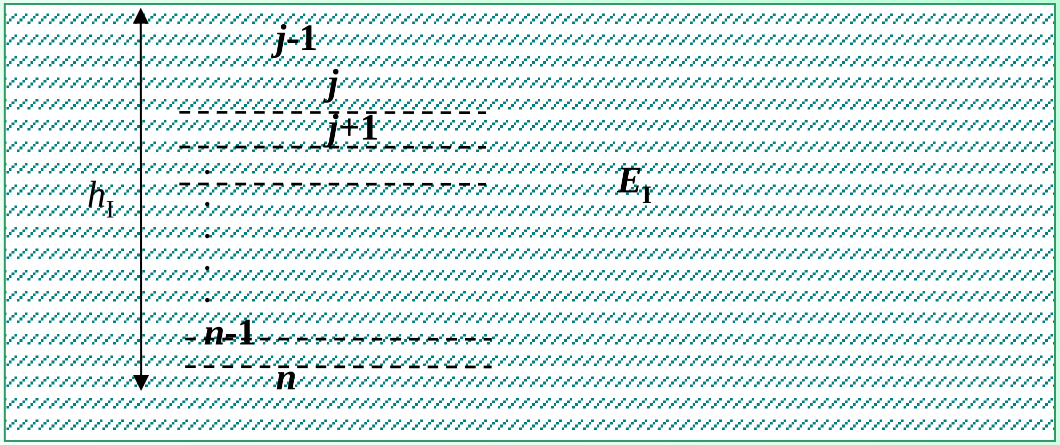
$$n_{SG} = \text{int}\left(\frac{h_{SG}}{12}\right) \text{for } h_{SB} > 305\,\text{mm}$$

Quadratic-Varied Modulus



$$\int_0^{h_I} E(z)dz = E_I h_I$$

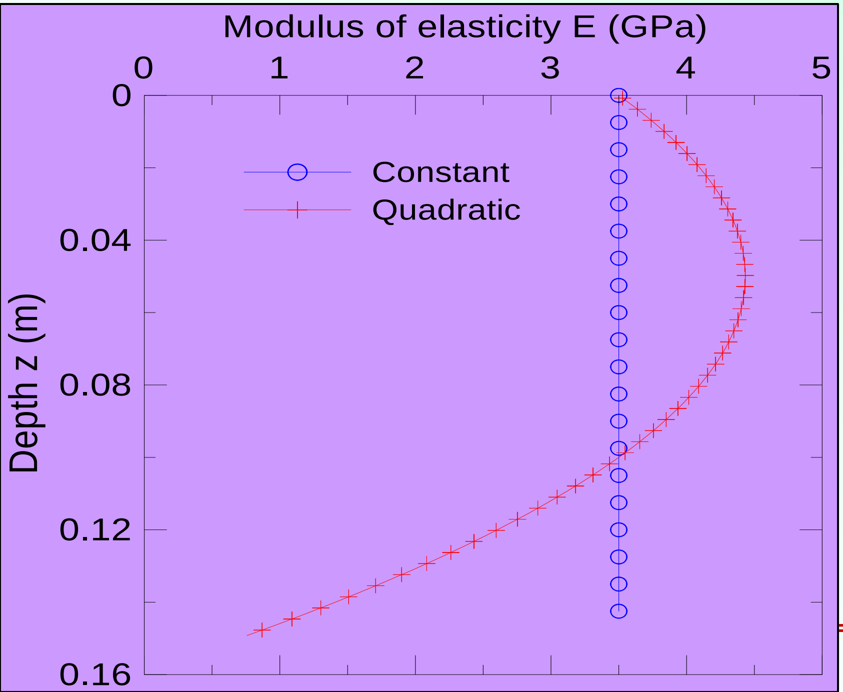
$$E_{j-1}/E_j \approx 0.90 - 1.00$$



Examples

Parameters of a typical flexible pavement example

Layer	Thickness (cm)	Resilient Modulus (MPa)	Poisson's Ratio
AC Layer	15	3500	0.3
Base Layer	25	700	0.3
Subbase Layer	25	300	0.3
Subgrade Layer	Infinite Half-Space	100	0.3

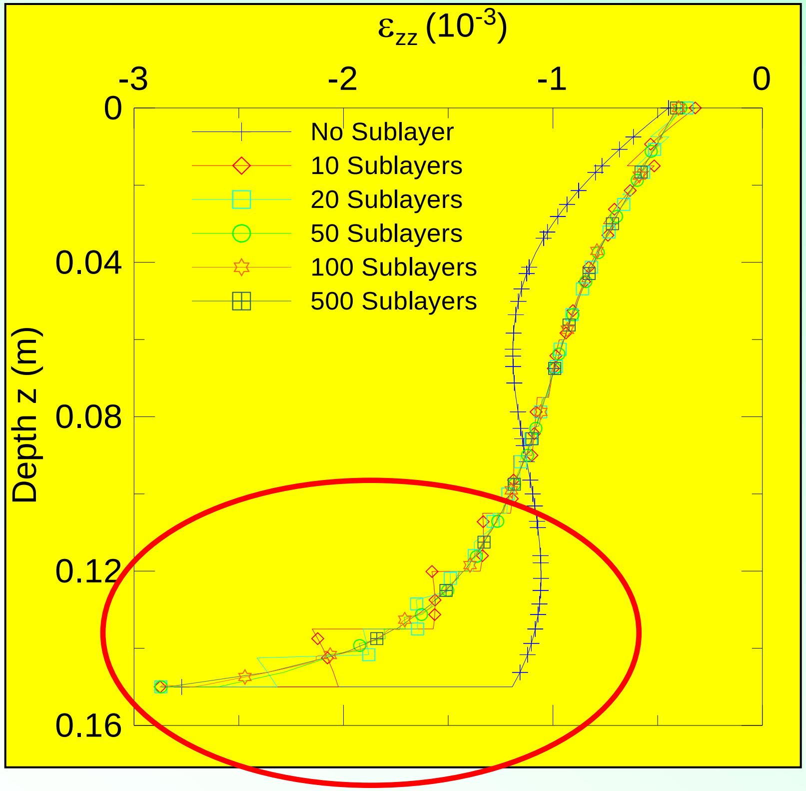


Modulus of elasticity variation with depth in the AC layer

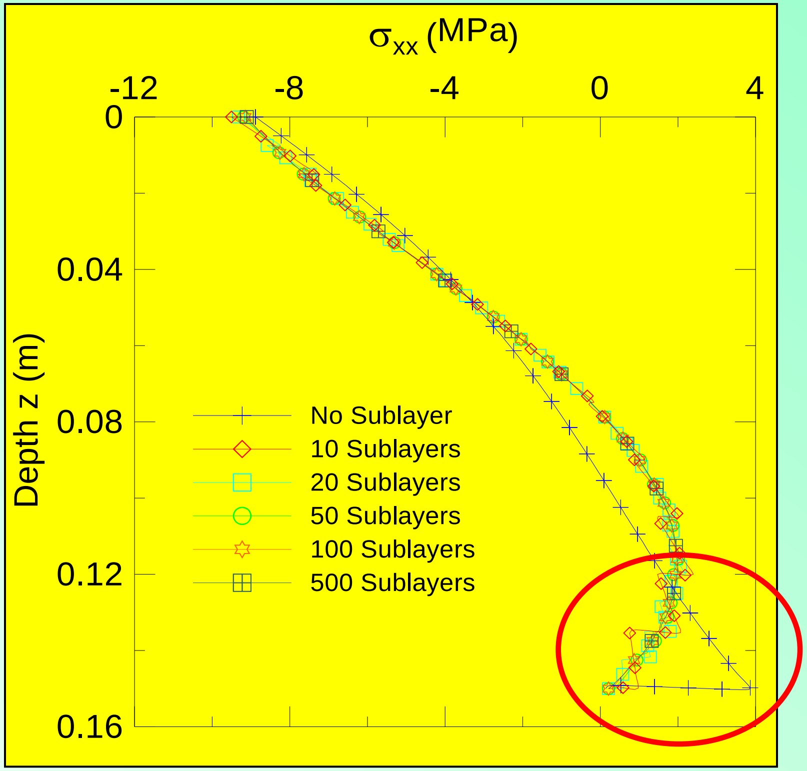
10 sublayers
20 sublayers
50 sublayers
100 sublayers
500 sublayers

MultiSmart 3D

Examples

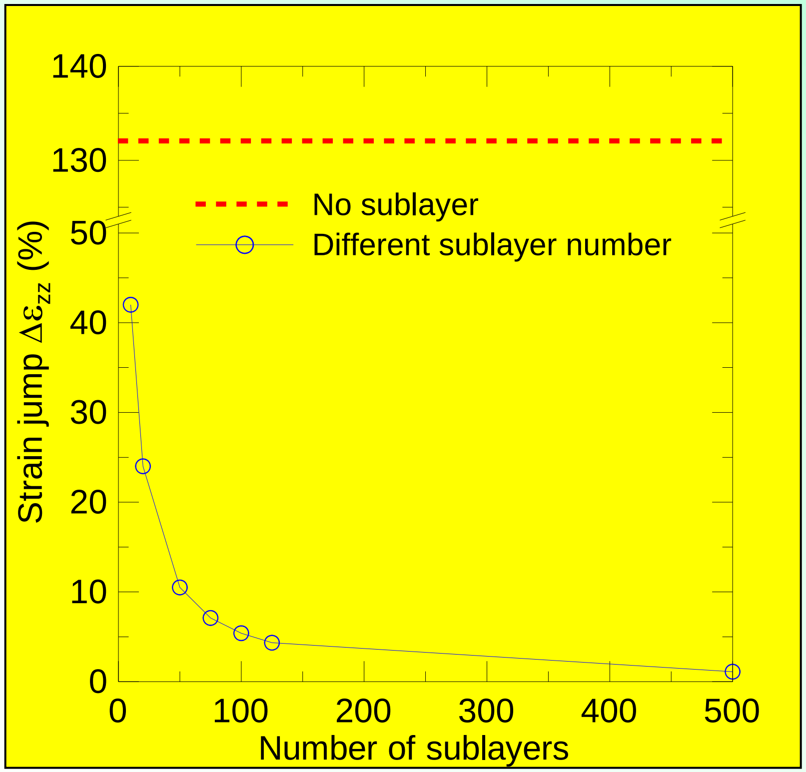


Vertical strain variation with depth in AC layer

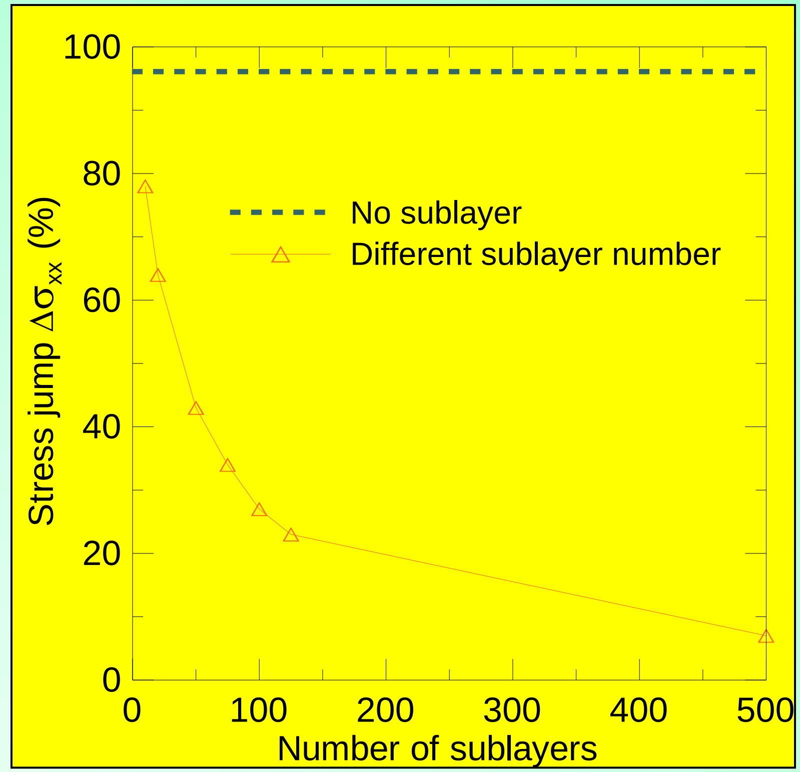


Horizontal stress variation with depth in AC layer

Examples (cont'd)

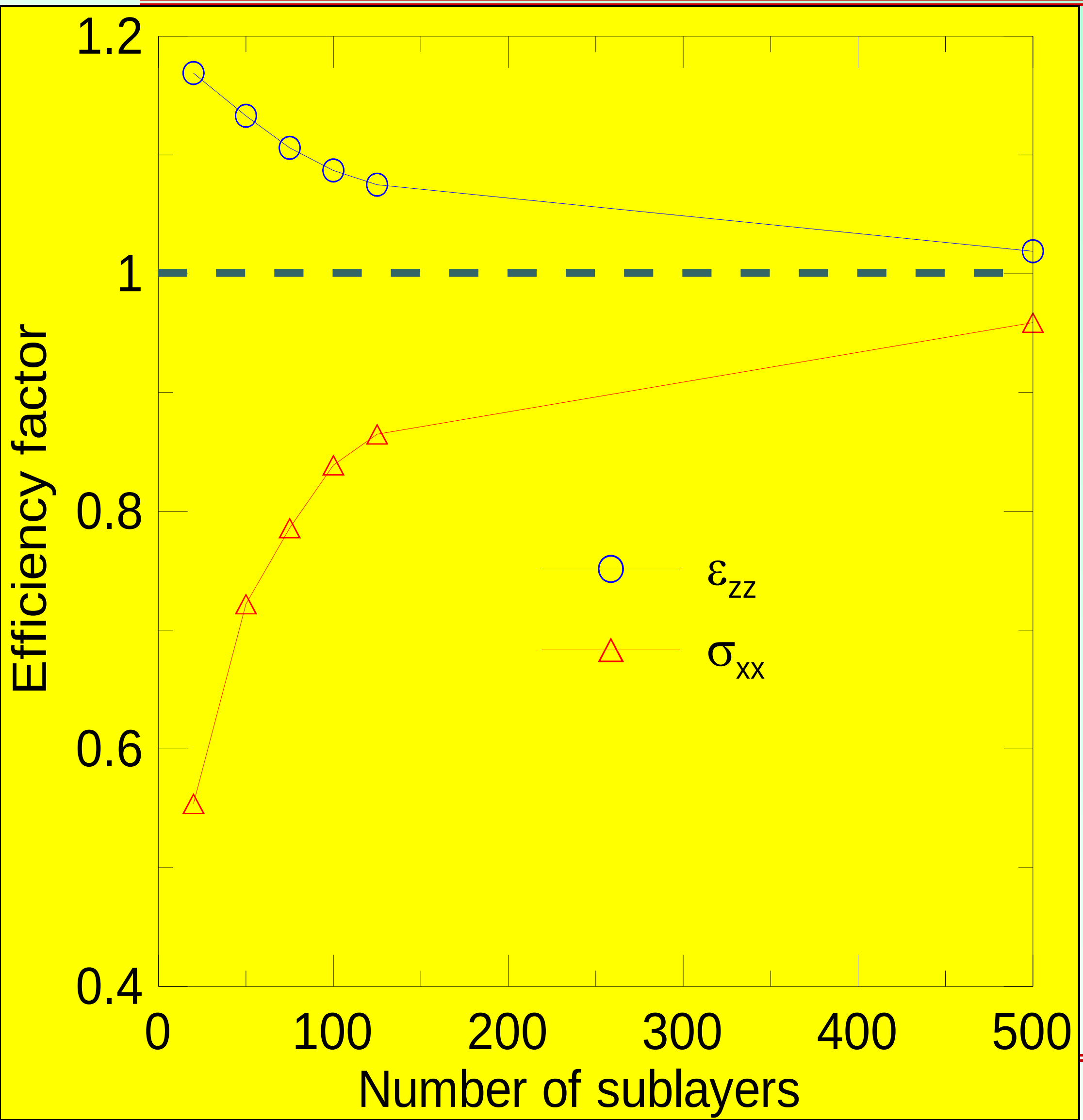


Vertical strain jumps at the interface with sublayer number



Horizontal stress jumps at the interface with sublayer number

Efficiency Factors

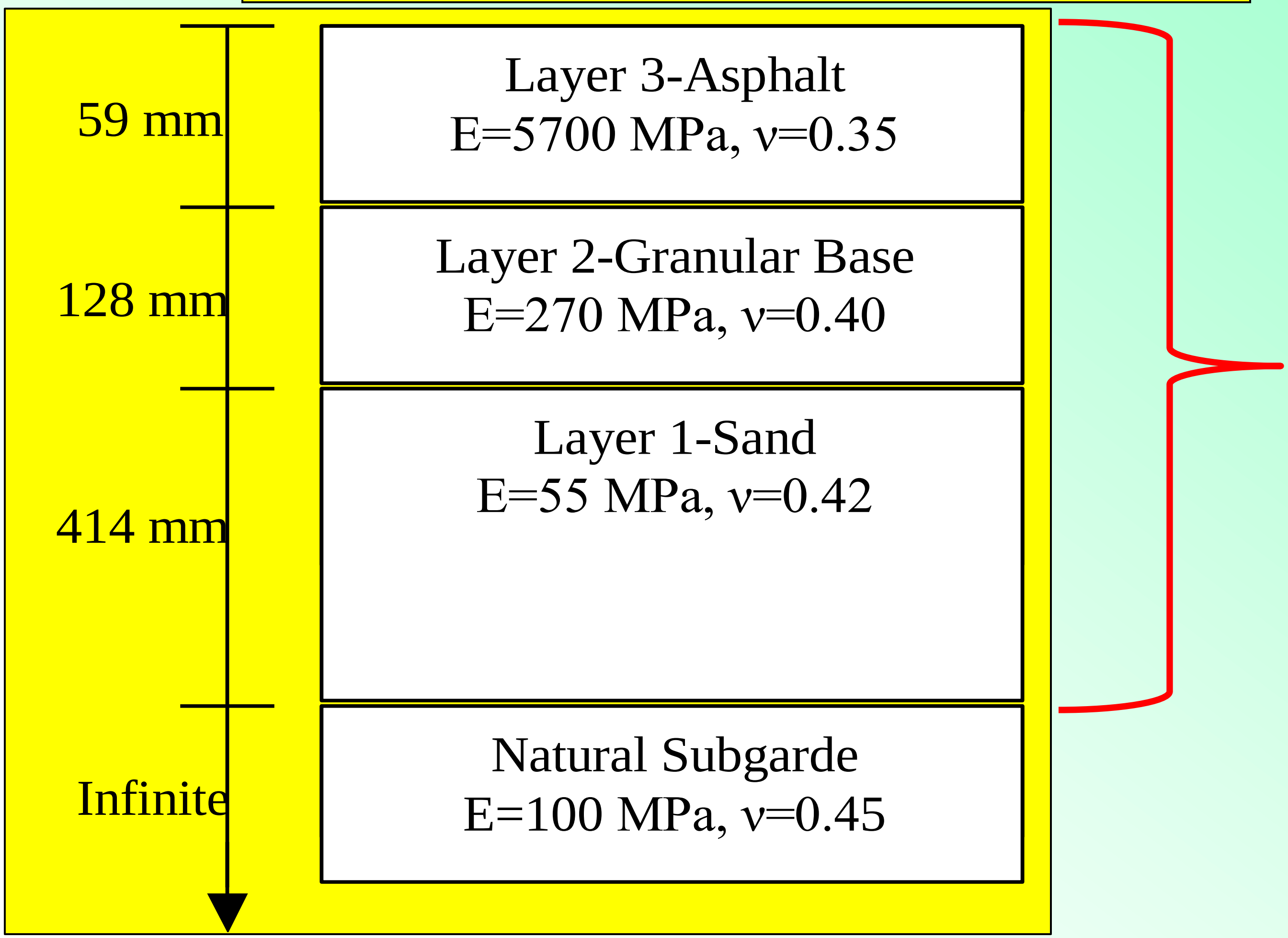


$$EF_{Stress} = \left(\frac{Stress_{j+1}}{Stress_{j}} \right) \Big/ \left(\frac{E_{j}}{E_{j+1}} \right)$$

$$EF_{Strain} = \left(\frac{Strain_{j+1}}{Strain_{j}} \right) \Big/ \left(\frac{E_{j}}{E_{j+1}} \right)$$

Verification of Quadratic-Varied Modulus

Pavement system at the test site after construction



1625 sublayers

$E_{j-1} / E_j \approx 0.97$

pressure 500 kPa

load radius 0.15m

Hildebrand (2002)

Verification of Quadratic-Varied Modulus

Instrument depth and location at the test site

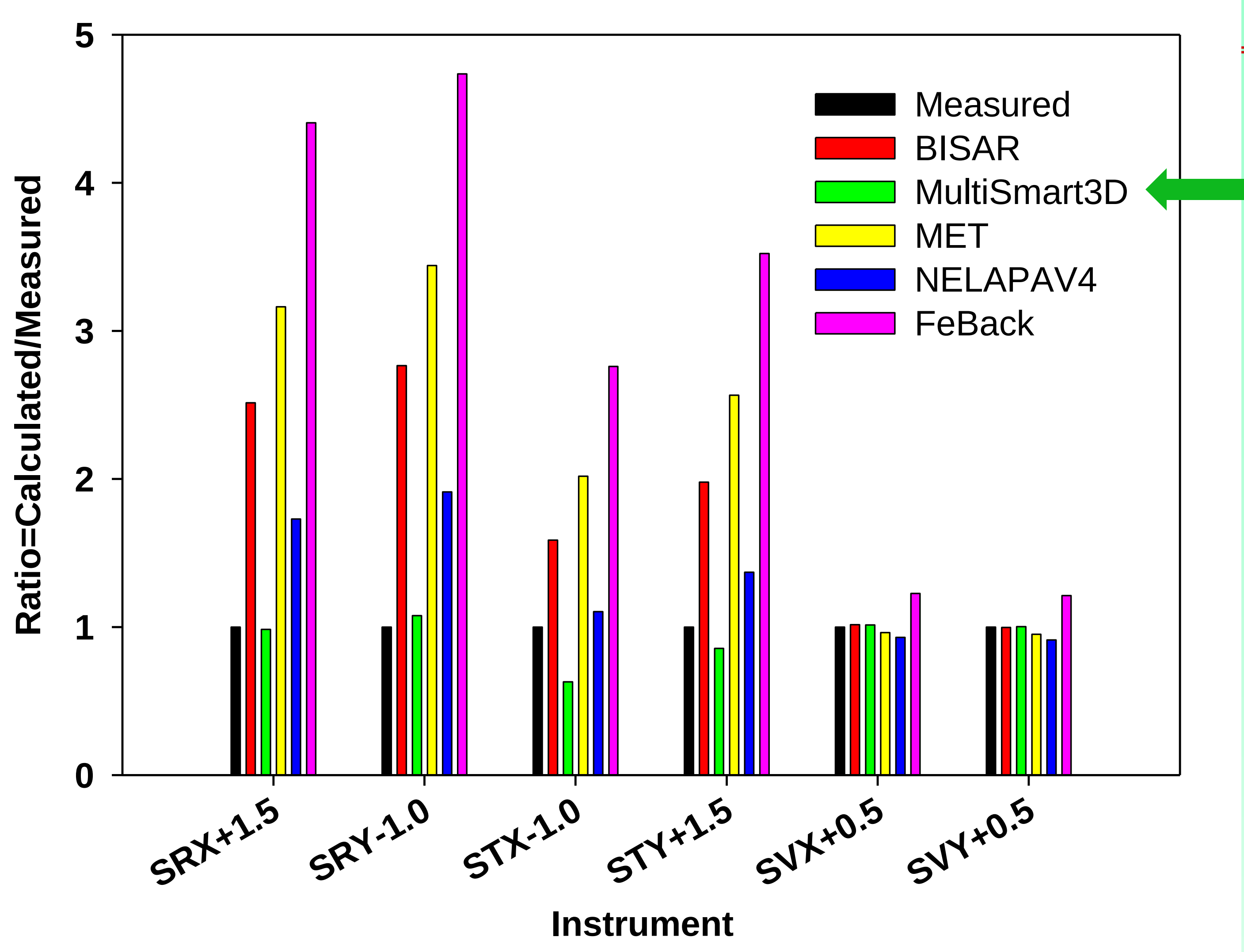
Response Type	Instrument ID	X (m)	Y (m)	Depth (m)
σ_{x*}	SRX+1.5	1.50	0.00	0.568
σ_{y*}	SRY-1.0	0.00	-1.00	0.571
σ_{y}	STX-1.0	-1.00	0.00	0.569
σ_{x}	STY+1.5	0.00	1.50	0.566
σ_{z*}	SVX+0.5	0.50	0.00	0.559
σ_{z}	SVY+0.5	0.00	0.50	0.559
ε_{x*}	TRX+1.0	1.00	0.00	0.538
ε_{y*}	TRY-1.5	0.00	-1.50	0.540
ε_{y}	TTX-1.5	-1.50	0.00	0.543
ε_{x}	TTY+1.0	0.00	1.00	0.539
ε_{z*}	TVX-0.5	-0.50	0.00	0.550
ε_{z}	TVY-0.5	0.00	-0.50	0.552

Verification of Quadratic-Varied Modulus

Stresses (kPa) at the actual instrument location

Instrument	Response	Measured	MultiSmart3D	BISAR	MET	NELAPAV4	FeBack
SRX+1.5	σ_{x*}	3.70	3.64	9.30	11.70	6.40	16.30
SRY-1.0	σ_{y*}	3.40	3.66	9.40	11.70	6.50	16.10
STX-1.0	σ_{y}	5.80	3.65	9.20	11.70	6.40	16.00
STY+1.5	σ_{x}	4.60	3.93	9.10	11.80	6.30	16.20
SVX+0.5	σ_{z*}	33.90	34.37	34.40	32.60	31.50	41.60
SVY+0.5	σ_{z}	34.30	34.37	34.20	32.60	31.30	41.60

Verification of Quadratic-Varied Modulus



Measured and calculated stress comparison

Conclusions

✐Modulus varies along depth and thus more sublayers are needed to model the modulus variation;

✐Average modulus of elasticity will cause jump in the stresses and strains at the interface while continuous variation of the modulus can reduce the jump;

✐The more the sublayers one has, the less the jump in the response; the more difficult in computation;

✐Moduli variation within any layer in the pavement system can be performed using a quadratic relation if no actual moduli variation data are available;

✐Efficiency factor was introduced to measure the needed number of sublayers and the effectiveness of the modulus variation with depth.

Outline

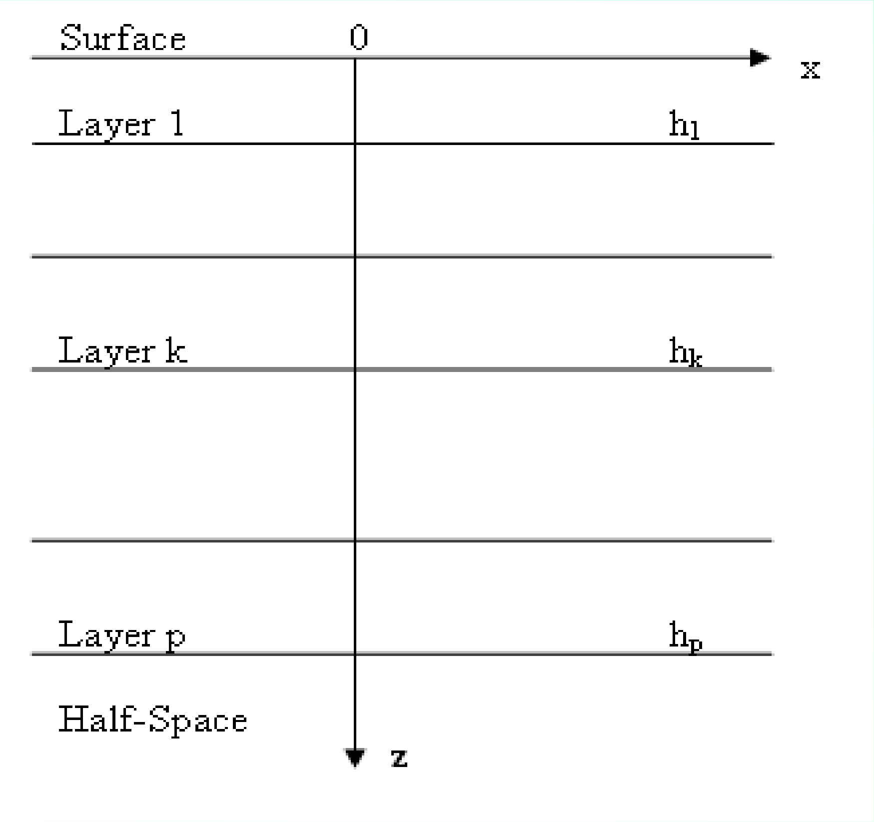
--- General Formulation and Solutions

--- Graded Moduli

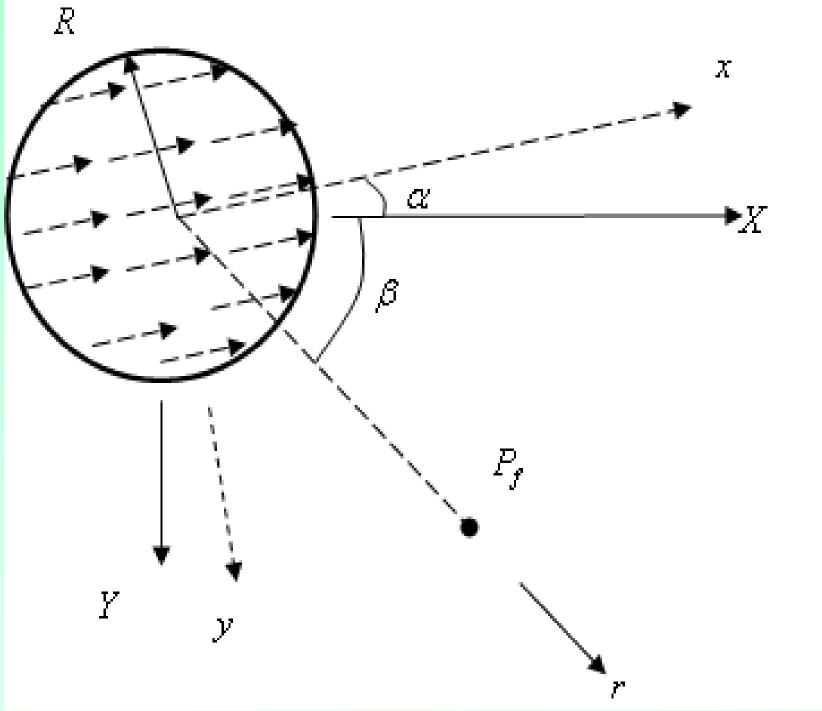
--- Transverse Isotropy/Shearing

--- Conclusions

Model Description



Geometry of a transversely isotropic multilayered half space



Geometry of surface loading and boundary conditions

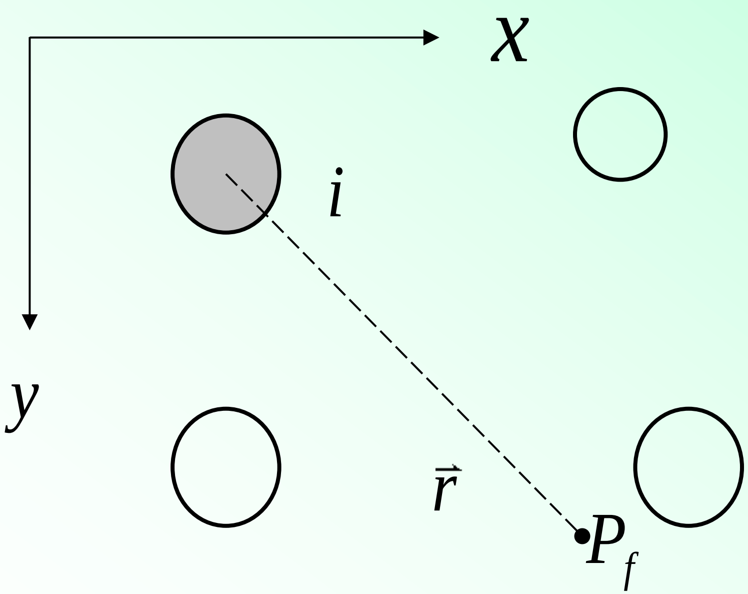
$$\begin{cases} \sigma_{rz}(r,\theta,0) = q\cos\theta \\ \sigma_{\theta z}(r,\theta,0) = -q\sin\theta \\ \sigma_{zz}(r,\theta,0) = 0 \end{cases} \qquad r < R$$

$$\theta = \beta - \alpha$$

Solutions under multiple loads

In global coordinates

Multiple loads



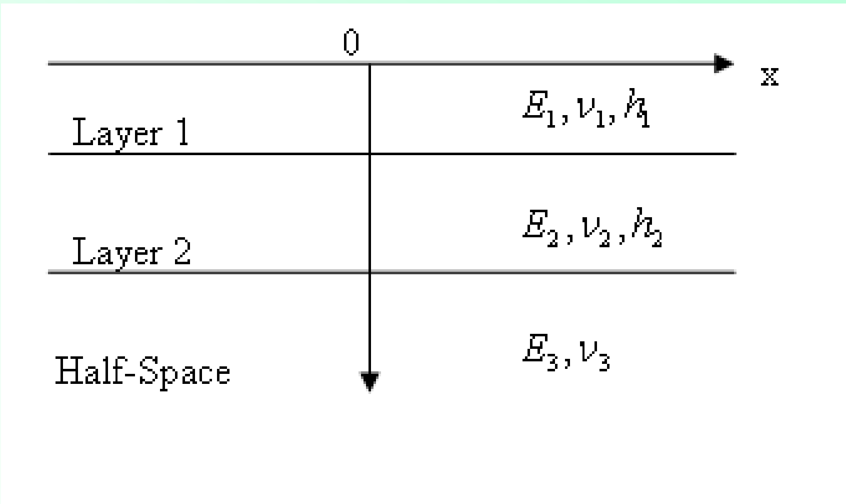
Superposition principle

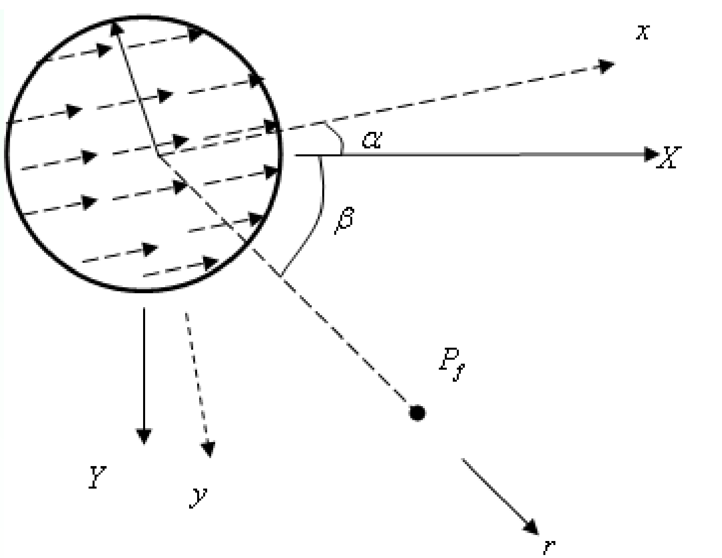
$$\begin{Bmatrix} u_x(P_f) \\ u_x(P_f) \\ u_z(P_f) \end{Bmatrix} = \sum_{i=1}^{N} [S_i]^T \begin{Bmatrix} u_r(P_f) \\ u_\theta(P_f) \\ u_z(P_f) \end{Bmatrix}_i$$

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ & \sigma_{yy} & \sigma_{yz} \\ & & \sigma_{zz} \end{bmatrix} = \sum_{i=1}^{N} [S_i]^T \begin{bmatrix} \sigma_{rr} & \sigma_{r\theta} & \sigma_{rz} \\ & \sigma_{\theta\theta} & \sigma_{\theta z} \\ & & \sigma_{zz} \end{bmatrix}_i [S_i]$$

$$[S_i] = \begin{bmatrix} \cos(\beta_i) & \sin(\beta_i) & 0 \\ -\sin(\beta_i) & \cos(\beta_i) & 0 \\ & & 1 \end{bmatrix}$$

Examples





$$E_1 = 2500MPa, v_1 = 0.35, h_1 = 10cm$$
$$E_2 = 280MPa, v_2 = 0.35, h_2 = 35cm$$
$$E_3 = 50MPa, v_3 = 0.4$$

single load

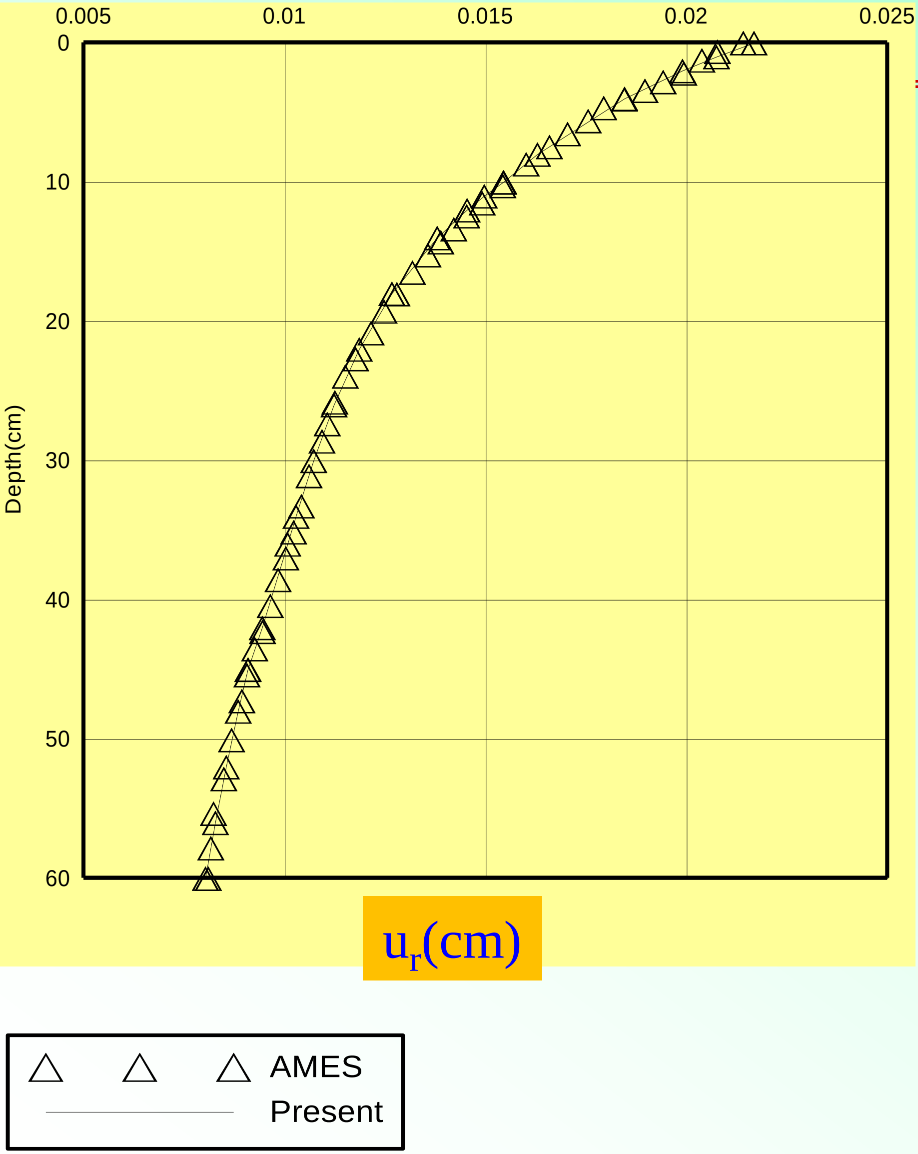
$$Q = 49kN, R = 15cm, \alpha = \pi / 6$$

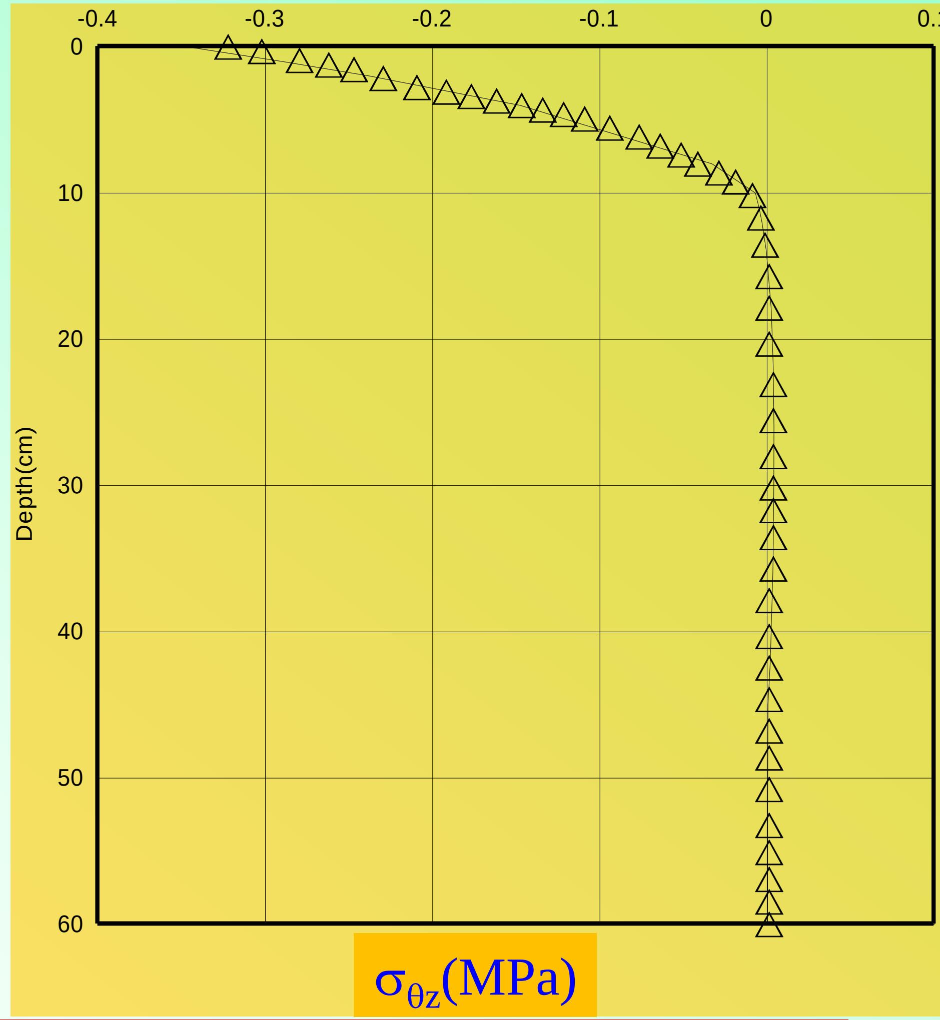
double loads

$$Q_1 = Q_2 = 49kN, R_1 = R_2 = 15cm$$
$$\alpha_1 = -\pi / 6, \alpha_2 = \pi / 6$$
$$O_1(0, -0.15), O_2(0, 0.15)$$

Displacements and Stresses

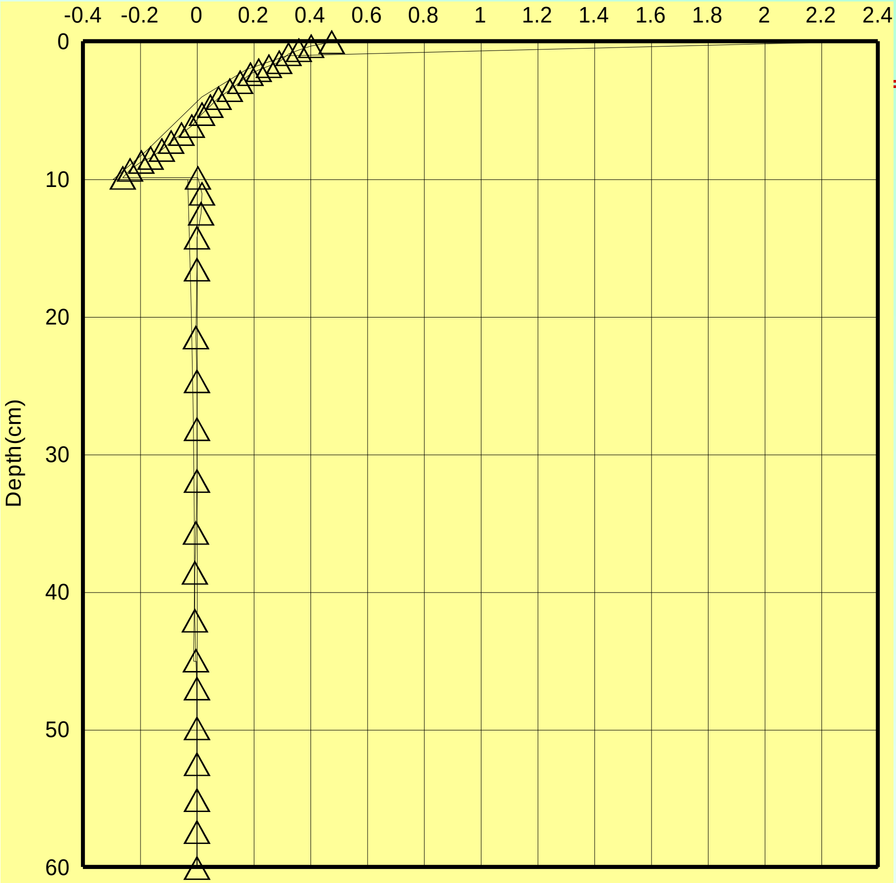
single load



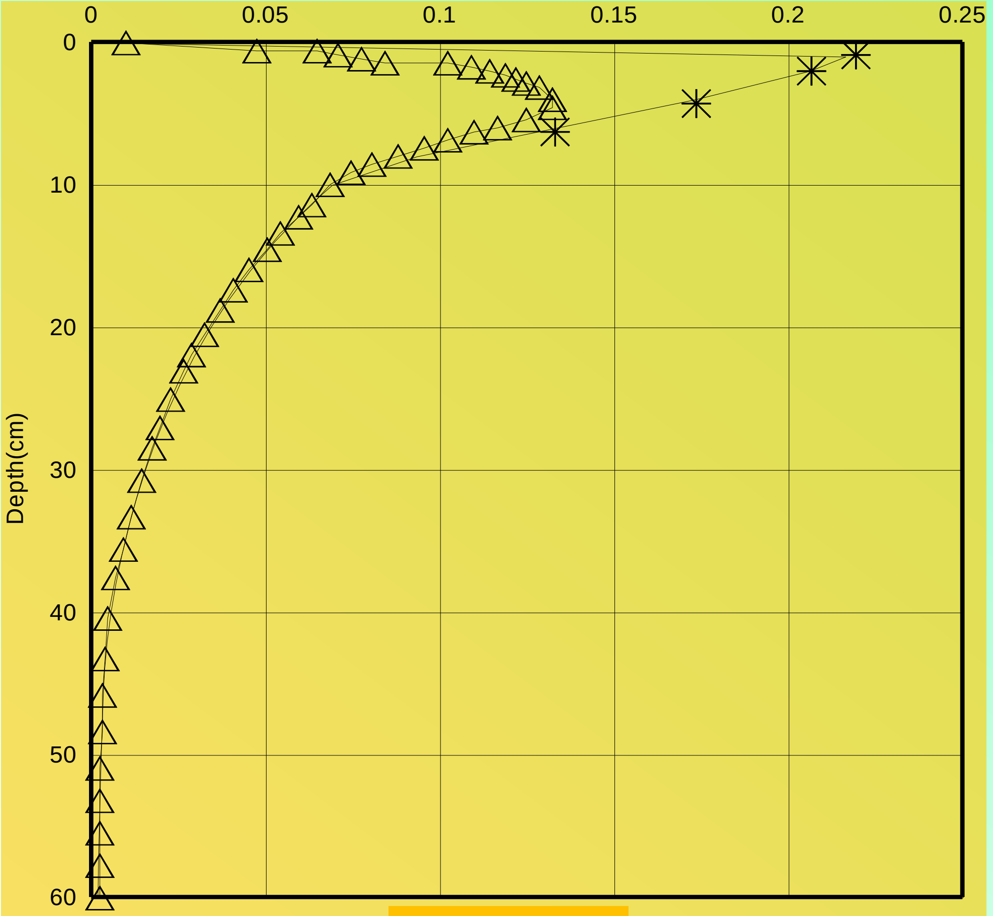


Displacements and Stresses

double loads



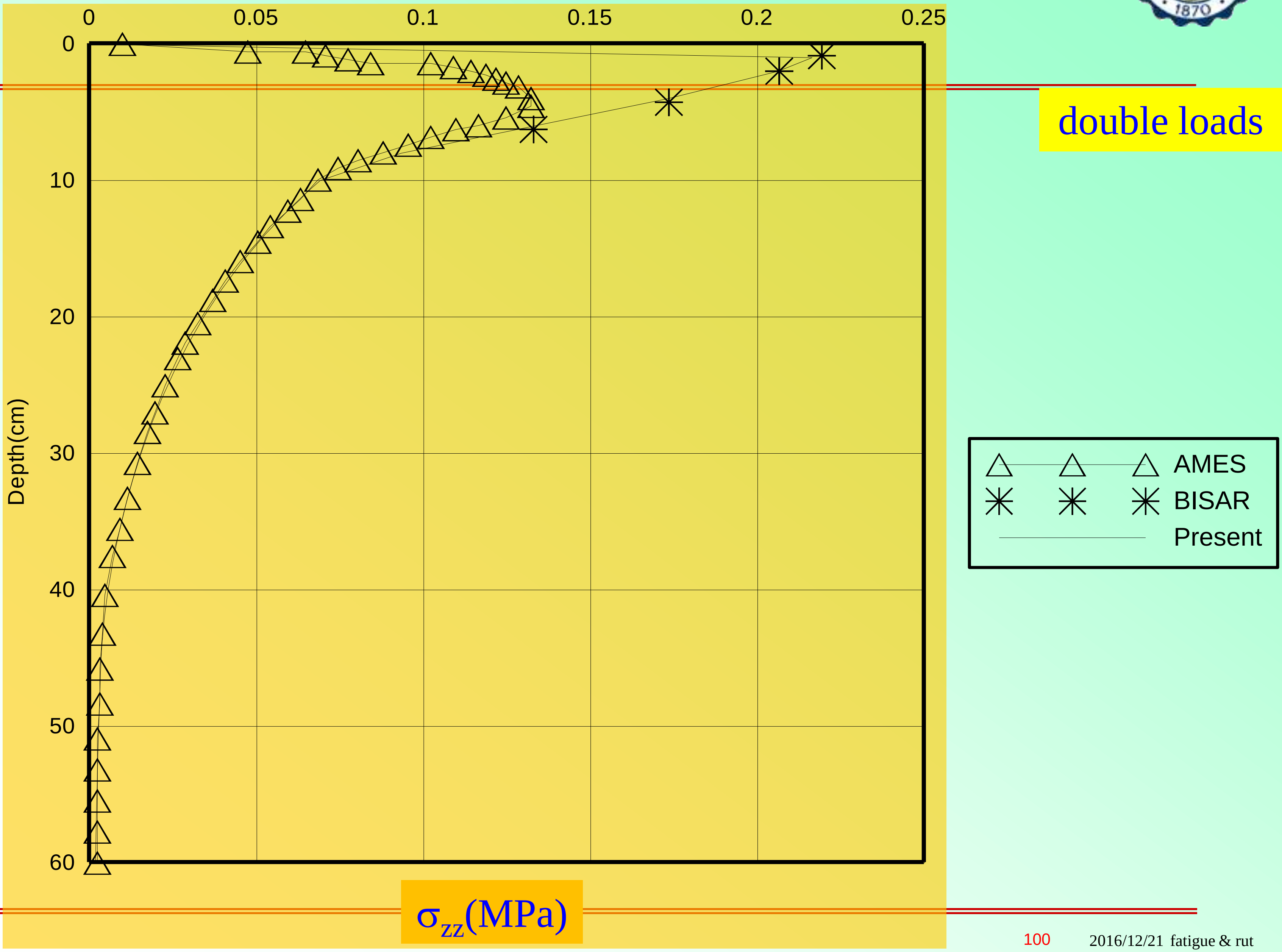
σ_{xx}(MPa)



σ_{zz}(MPa)

△ △ △ AMES
Present

Stress Comparison



double loads

σ_{zz}(MPa)

Impact to Pavement Design - Models

Fatigue

$$N_f = A_f K F'' \left(\frac{1}{\varepsilon_t}\right)^5 E_s^{-1.4}$$

Shell Model

N_f : the number of load repetitions to fatigue cracking
F'': A constant that depends on the layer thickness and the material stiffness.
ε_t : the tensile strain at the bottom of the AC layer
E_s: the stiffness of the material
A_f and K : material constants

$$N_f = 0.00432C\left(\frac{1}{\varepsilon_t}\right)^{3.291}\left(\frac{1}{E_s}\right)^{0.854}$$

Asphalt Institute Model

N_f : the number of load repetitions to fatigue cracking
ε_t : the tensile strain at the bottom of the AC layer
E_s: the stiffness of the material
C : material constant

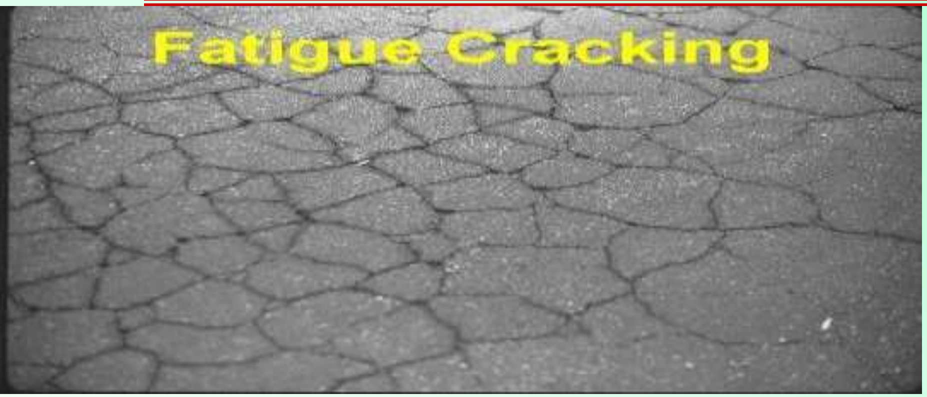
Rutting

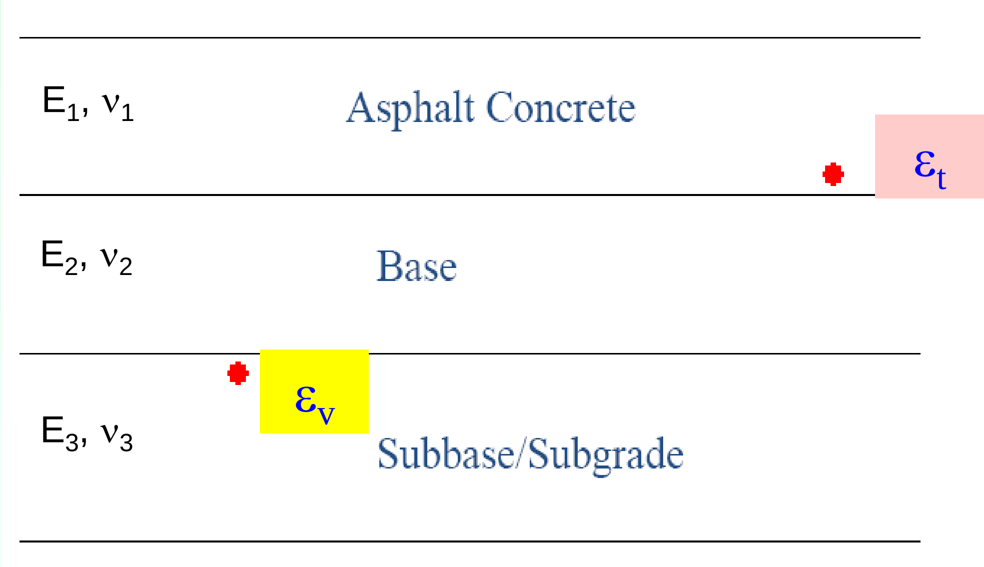
$$N_r = (5.5).10^{15}\left(\frac{1}{\varepsilon_v}\right)^{3.929}$$

MnROAD Rutting Model

N_r : the number of allowable load repetitions until rutting failure
ε_v : the maximum compressive strain at the top of the subgrade layer.

Pavement Failure





$$N_f = 2.83 \times 10^{-6} \left(\frac{10^6}{\varepsilon_t}\right)^{3.148}$$

$$N_r = 1 \times 10^{16} \left(\frac{1}{\varepsilon_v}\right)^{3.87}$$

(Timm & Newcomb, 2006)

Horizontal vs. Vertical Loads (1)

$E_1 = 2500MPa, v_1 = 0.35, h_1 = 10cm$

$E_2 = 280MPa, v_2 = 0.35, h_2 = 35cm$

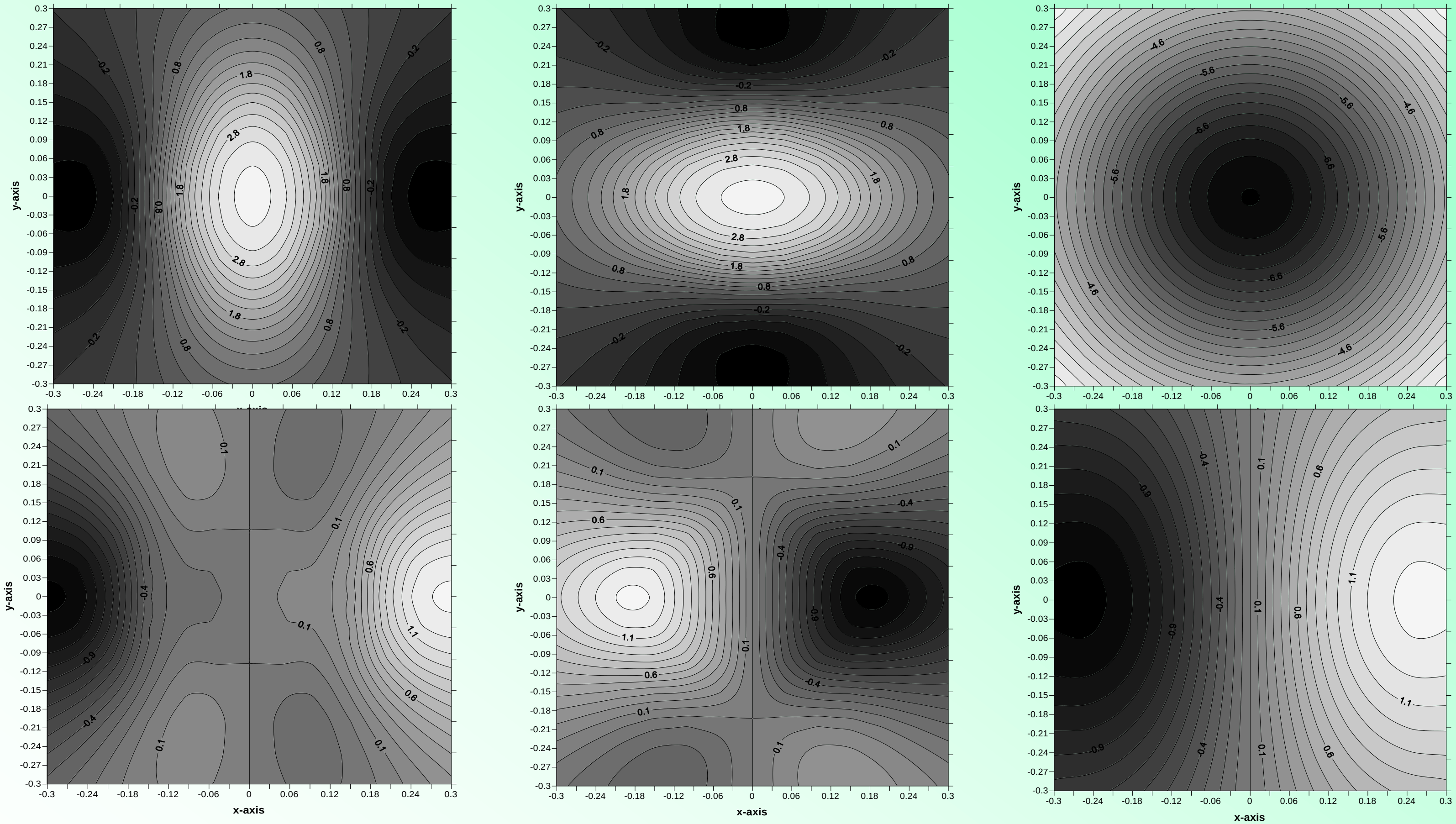
$E_3 = 50MPa, v_3 = 0.4$

0.6m×06m

single load

Vertical

Horizontal



$\varepsilon_{xx}(\times 10^{-4}$, z=0.0999m)

$\varepsilon_{yy}(\times 10^{-4}$, z=0.0999)

$\varepsilon_{zz}(\times 10^{-4}$, z=0.4501)

Horizontal vs. Vertical Loads (2)

Effect of horizontal load on fatigue and rutting performance

single load

$Q = 49kN, R = 15cm, \alpha = \pi / 6$

b = shearing over vertical loads

$E_1 = 2500MPa, v_1 = 0.35, h_1 = 10cm$

$E_2 = 280MPa, v_2 = 0.35, h_2 = 35cm$

$E_3 = 50MPa, v_3 = 0.4$

	Fatigue				Rutting	
	ε_{xx} ($\times 10^{-4}$)	ε_{yy} ($\times 10^{-4}$)	ε_t ($\times 10^{-4}$)	N_f ($\times 10^{24}$)	ε_v ($\times 10^{-4}$)	N_r ($\times 10^{28}$)
$b = 0$	3.3	3.3	3.3	1.9929	-7.4	1.3064
$b = 0.25$	3.4	3.4	3.4	1.8142	-7.6	1.1783
$b = 1$	3.4	4.0	4.0	1.0877	-7.8	1.0660

Anisotropy parameters

$$A_{11} = \frac{E(1-\frac{E}{E'}\nu'^2)}{(1+\nu)(1-\nu-\frac{2E}{E'}\nu'^2)} \qquad A_{13} = \frac{E\nu'}{(1-\nu-\frac{2E}{E'}\nu'^2)} \qquad A_{44} = G'$$

$$A_{33} = \frac{E'(1-\nu)}{(1-\nu-\frac{2E}{E'}\nu'^2)} \qquad A_{66} = \frac{E}{2(1+\nu)} \qquad A_{12} = A_{11} - 2A_{66}$$

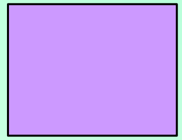
E, E' Young's moduli in the plane of transverse isotropy and a direction normal to it, respectively

ν, ν' Poisson's ratios characterizing the lateral strain responses in the plane of transverse isotropy to a stress acting parallel or normal to it, respectively

G' Shear modulus in planes normal to the plane of transverse isotropy

(Wardle, 1981)

Anisotropy Effect (1)

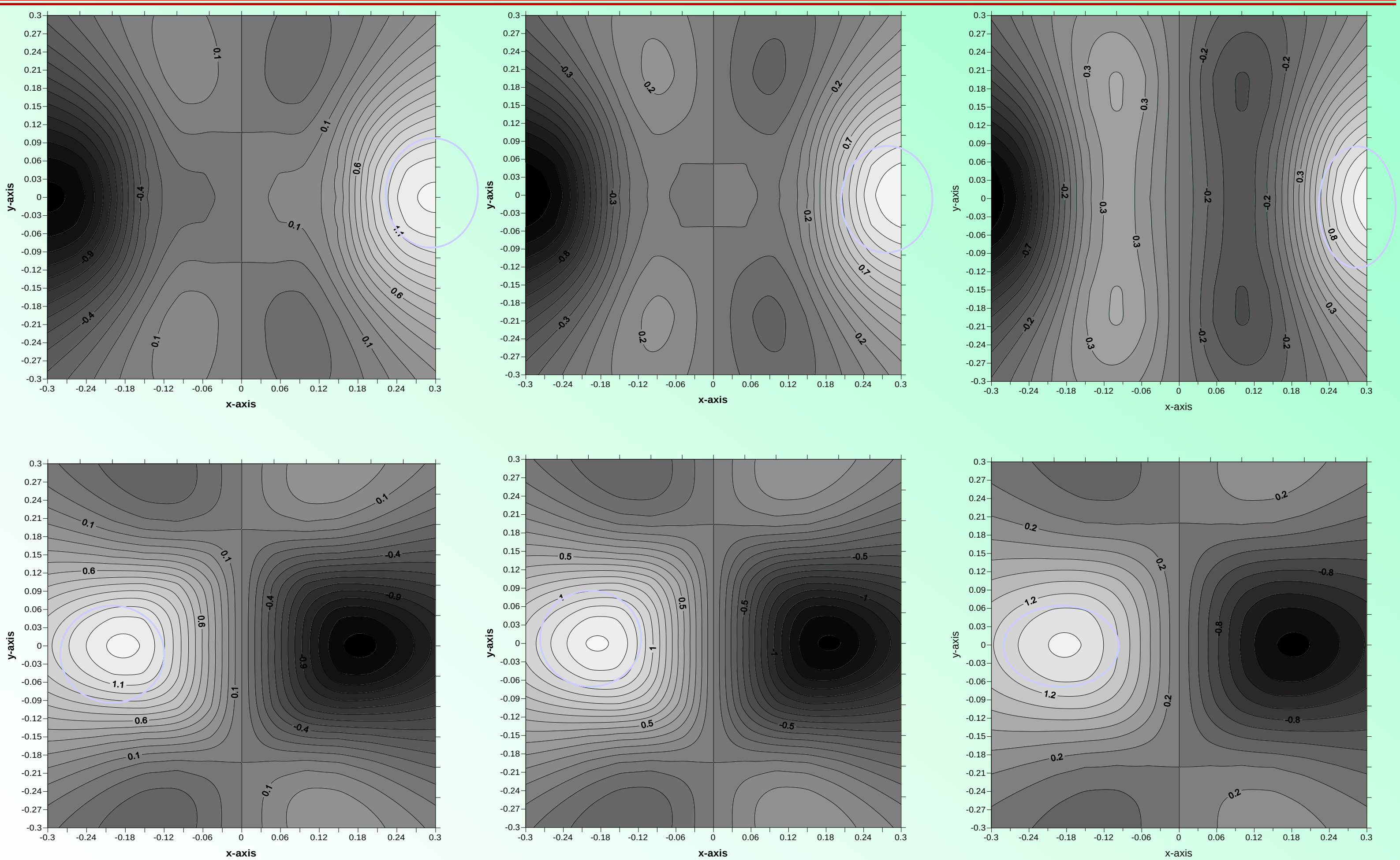


0.6m×06m

Single shear load

$\varepsilon_{xx}(\times 10^{-4})$

$\varepsilon_{yy}(\times 10^{-4})$



$E/E'=1.0$ $E/E'=1.5$ $E/E'=3.0$

Anisotropy Effect (2)

Single shear load

Effect of anisotropy on fatigue

Case 1
Case 2
Case 3

Case 1
Case 4
Case 5

Case 1
Case 6
Case 7

	E/E'	G/G'	ν/ν'	ε_{xx} ($\times10^{-4}$)	ε_{yy} ($\times10^{-4}$)	ε_t ($\times10^{-4}$)
Case 1	**1.0**	**1.0**	**1.0**	**1.4**	**1.4**	1.4
Case 2	**1.5**	1.0	1.0	1.3	**1.5**	1.5
Case 3	**3.0**	1.0	1.0	1.2	**1.8**	1.8
Case 4	1.0	**2.0**	1.0	1.2	**1.3**	1.3
Case 5	1.0	**3.0**	1.0	1.0	**1.2**	1.2
Case 6	1.0	1.0	**0.75**	**1.4**	1.4	1.4
Case 7	1.0	1.0	**1.5**	**1.4**	1.3	1.4

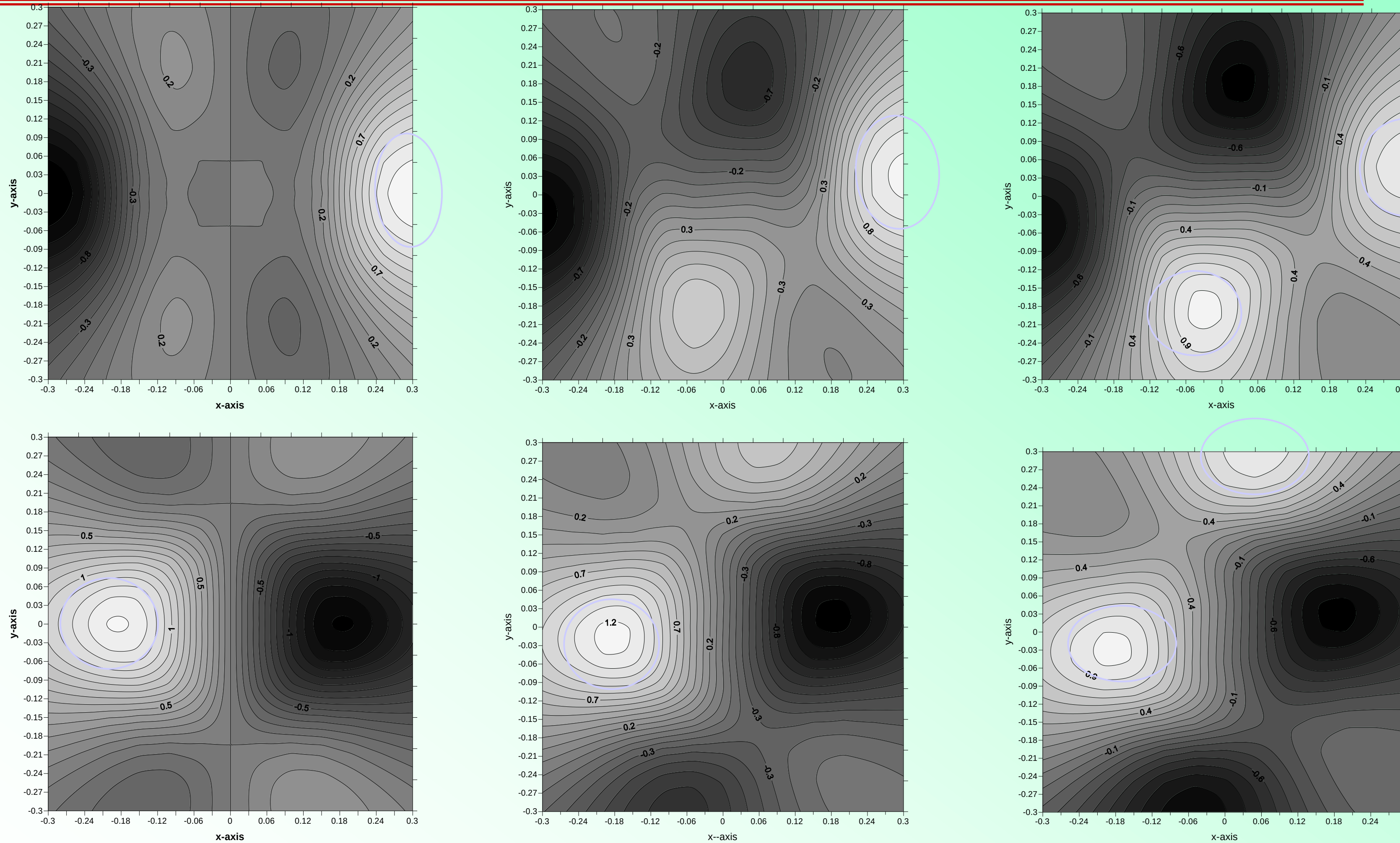
Orientation Effect (1)

0.6m×06m

Single shear load

$\varepsilon_{xx}(\times 10^{-4})$

$\varepsilon_{yy}(\times 10^{-4})$



$\alpha = 0^{\circ}$ $\alpha = 30^{\circ}$ $\alpha = 45^{\circ}$

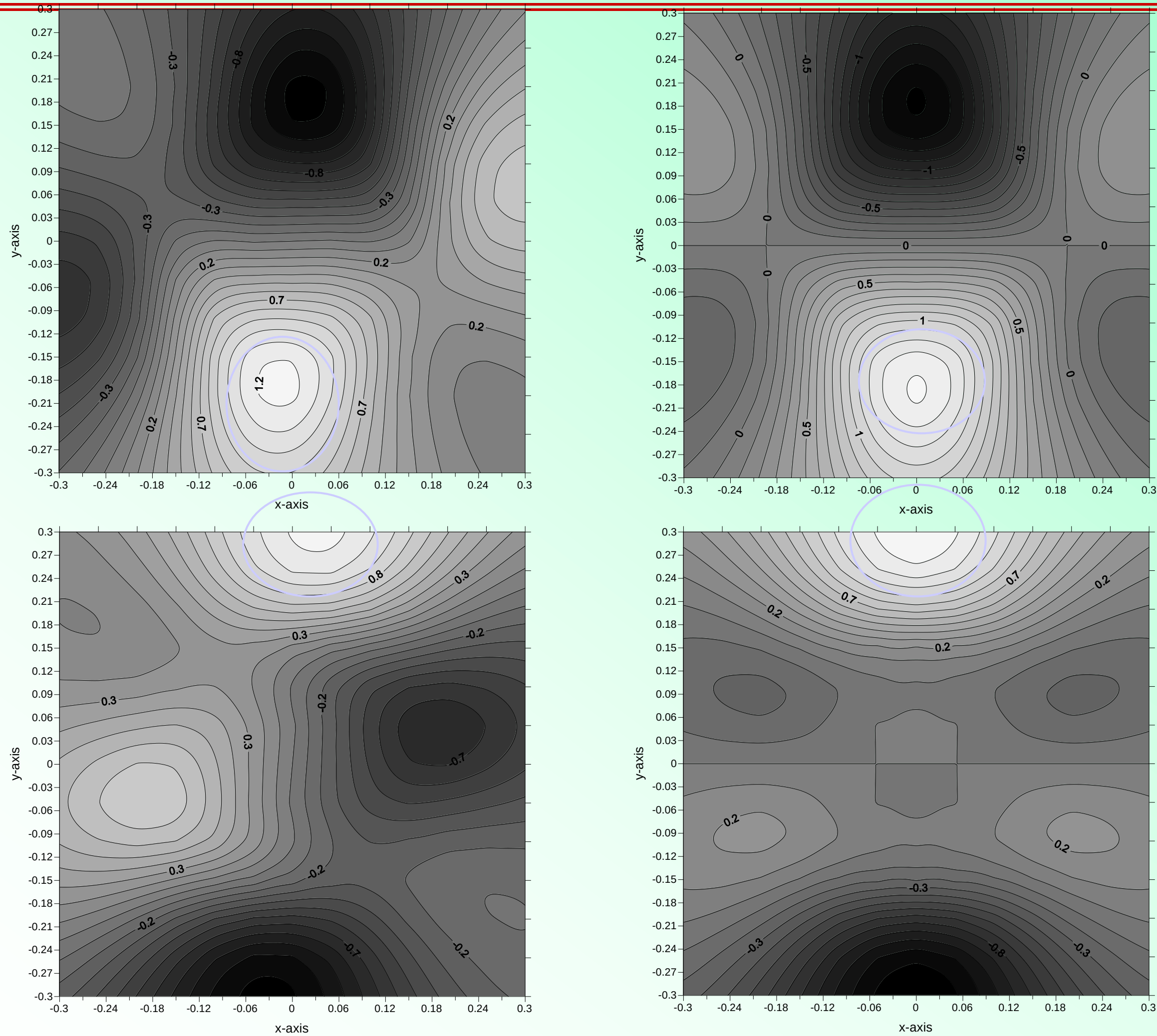
Orientation Effect (2)

0.6m×06m

Single shear load

$\varepsilon_{xx}(\times 10^{-4})$

$\varepsilon_{yy}(\times 10^{-4})$



$\alpha = 60^\circ$

$\alpha = 90^\circ$

Effect of loading Orientations

	ε_{xx}	ε_{yy}	ε_t
$\alpha = 0^\circ$	1.4	1.4	1.4
$\alpha = 30^\circ$	1.2	1.3	1.3
$\alpha = 45^\circ$	1.1	1.1	1.1
$\alpha = 60^\circ$	1.3	1.2	1.3
$\alpha = 90^\circ$	1.4	1.4	1.4

Conclusions

- The solutions for transversely isotropic layered half-space under horizontal load are established, and a reliable program is developed;
- Ignoring the horizontal load will over-estimate the lifetime expectance of pavement greatly;
- Anisotropy will change greatly the magnitude of critical strain but slightly the position;
- Critical strains are most sensitive to *E*, then to *G*, less to nu;
- Load orientation will change both the magnitude and the position of critical strain.

Outline

--- General Formulation and Solutions

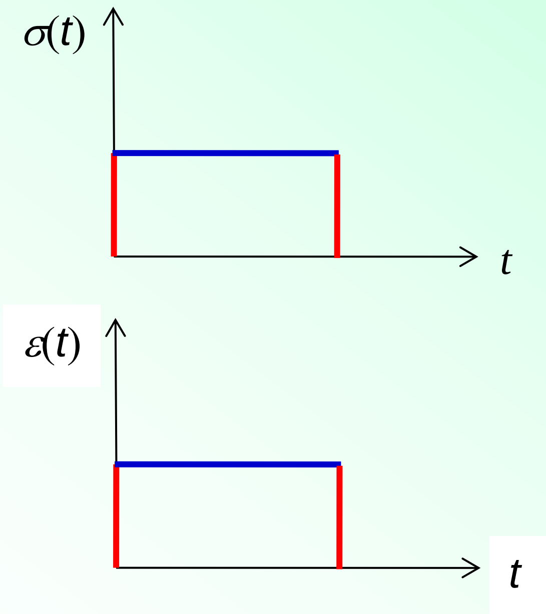
--- Graded Moduli

--- Transverse Isotropy/Shearing

--- Conclusions

Elasticity vs. Viscoelasticity

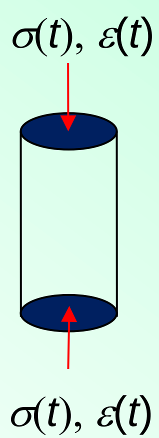
Elasticity



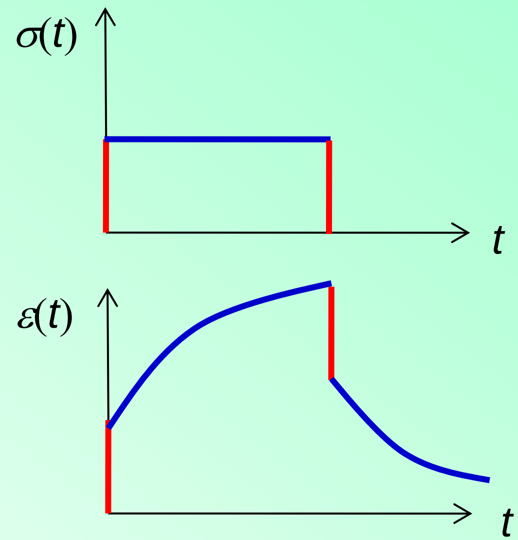
$$\varepsilon(t) = D\sigma(t)$$

$$\sigma(t) = E\varepsilon(t)$$

$$ED = 1$$



Viscoelasticity



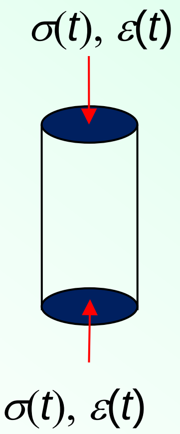
$$\varepsilon(t) = D(t) * \frac{d\sigma(t)}{dt}$$

$$\sigma(t) = E(t) * \frac{d\varepsilon(t)}{dt}$$

$$E(t) * D(t) = t$$

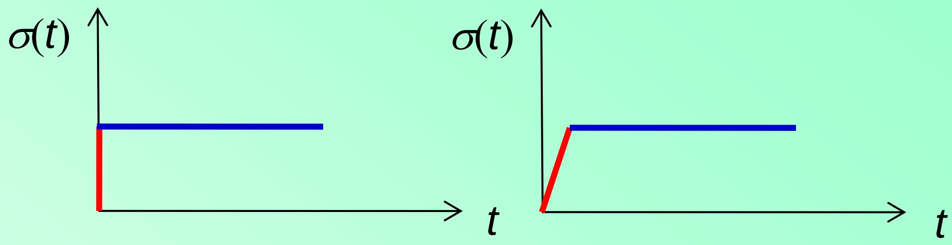
$$(f * g)(t) = \int_{-\infty}^{t} f(t-\tau)g(\tau)d\tau$$

Dynamic Modulus



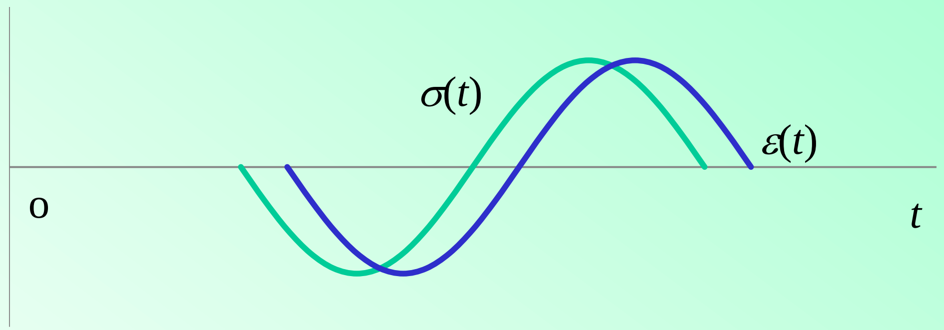
Step Load

(Monismith and Secor, 1962)



Oscilating Load

(Papazian 1962)



$$\sigma(t) = \sigma_0 e^{i\omega_0 t}$$

$$\sigma(t) = E(t) * \frac{d\varepsilon(t)}{dt}$$

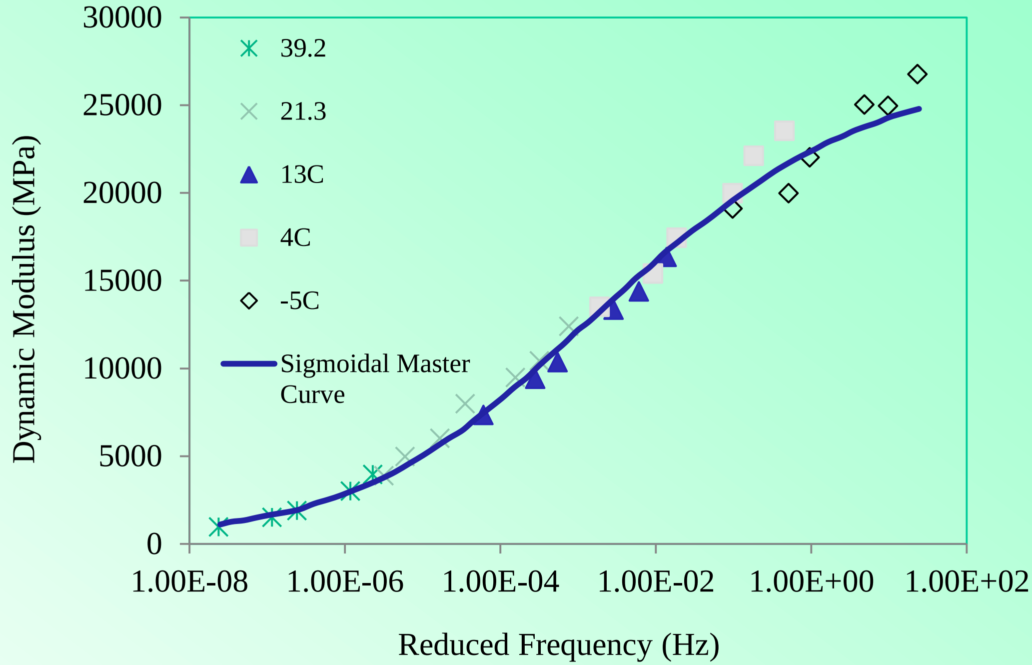
$$\varepsilon(t) = \varepsilon_0 e^{i\omega_0 t} e^{-i\Delta}$$

$$E^*(\omega_0) = \frac{\sigma_0}{\varepsilon_0} e^{i\Delta} = \frac{E(t) * \frac{de^{i\omega_0 t}}{dt}}{e^{i\omega_0 t}}$$

$$\left|E^*(\omega_0)\right| = \frac{\sigma_0}{\varepsilon_0} = \left|\frac{E(t) * \frac{de^{i\omega_0 t}}{dt}}{e^{i\omega_0 t}}\right|$$

Dynamic Modulus Test

AASHTO TP62-03 (2005)



Prony Series

$$D(t) = D_0 + \sum_{m=1}^{M} D_m (1 - e^{-t/\tau_m})$$

$$E(t) = E_e + \sum_{m=1}^{M} E_m e^{-t/\rho_m}$$

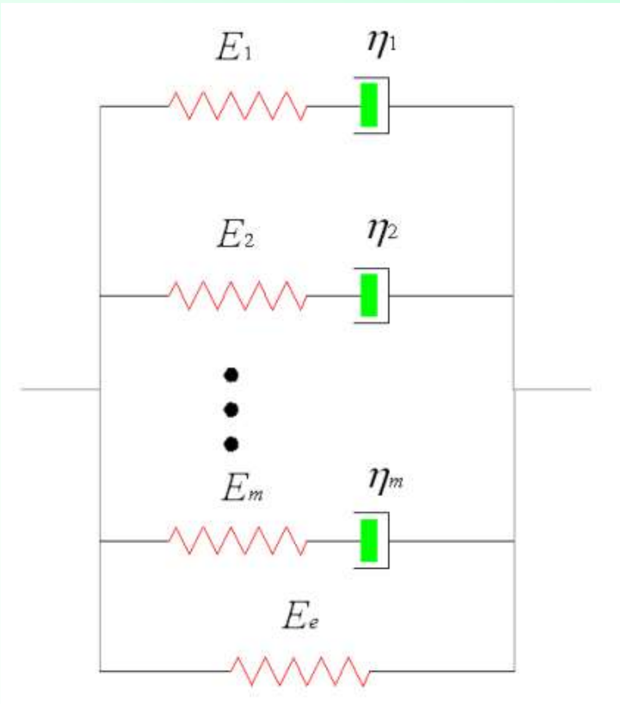
$$E(t) * D(t) = t$$

$$|E^*(\omega_0)| = \frac{\sigma_0}{\varepsilon_0} = \left| \frac{E(t) * \frac{de^{i\omega_0 t}}{dt}}{e^{i\omega_0 t}} \right|$$

Interconversion of Material Properties

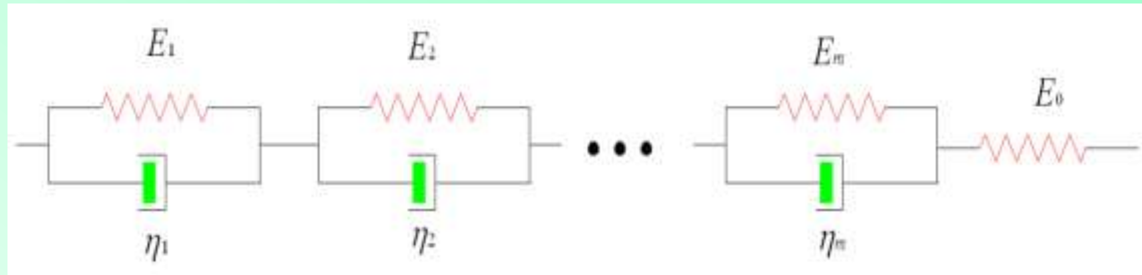
Mechanical Representation

Generalized Maxwell Model



$$E(t) = E_e + \sum_{m=1}^{M} E_m e^{-t/\rho_m}$$

Generalized Kelvin-Voigt Model



$$D(t) = \frac{1}{E_0} + \sum_{n=1}^{N} \frac{1}{E_n}(1 - e^{-E_n t/\zeta_n})$$

$$E(t) * D(t) = t$$

$$D(t) = D_0 + \sum_{m=1}^{M} D_m (1 - e^{-t/\tau_m})$$

Prony Series

Thermodynamic Representation

$$\frac{\partial W}{\partial q_i^{ex}} + \frac{\partial D}{\partial \dot{q}_i^{ex}} = Q_i^{ex} \qquad \frac{\partial W}{\partial q_i^{in}} + \frac{\partial D}{\partial \dot{q}_i^{in}} = Q_i^{in} = 0$$

W: Free energy
D: Dissipation function
q: state variable
Q: generalized force
ex: external variable
in: internal variable

$$W = \frac{1}{2}\sum a_{ij} q_i q_j \qquad D = \frac{1}{2}\sum b_{ij} \dot{q}_i \dot{q}_j$$

$$\sum_j \left(a_{ij} + b_{ij}\frac{\partial}{\partial t} \right) q_j = Q_i \quad \Rightarrow \quad \sum_j \left(a_{ij} + b_{ij} s \right) \tilde{q}_j = \tilde{Q}_i$$

$$\begin{bmatrix} N & \cdots & S \\ \vdots & & \vdots \\ T & \cdots & M \end{bmatrix} \begin{bmatrix} \tilde{q}_1 \\ \vdots \\ \tilde{q}_k \\ \tilde{q}_{k+1} \\ \vdots \\ \tilde{q}_n \end{bmatrix} = \begin{bmatrix} \tilde{Q}_1 \\ \vdots \\ \tilde{Q}_k \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\{\tilde{Q}_i\} = \left(N - SM^{-1}T \right)\{\tilde{q}_i\} \quad i = 1, \ldots, k$$

$$\tilde{Q}_i = \sum_{j=1}^{k} s \tilde{T}_{ij} \tilde{q}_j$$

$$\tilde{T}_{ij} = \sum_p \frac{1}{s + r_p} D_{ij}^{(p)} + \frac{D_{ij}}{s} + D_{ij}^{'}$$

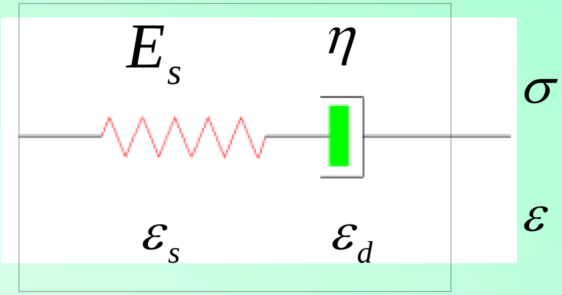
$$Q_i = \sum_{j=1}^{k} T_{ij} * \frac{dq_j}{dt}$$

$$T_{ij} = \sum_p e^{-r_p t} D_{ij}^{(p)} + D_{ij} + D_{ij}^{'}\delta(t)$$

Biot 1954, *J. Appl. Phys.*

Eigenvalue problem $[M]$

Thermodynamic Representation-Example



$$\varepsilon = \varepsilon_s + \varepsilon_d$$

$$\sigma = \varepsilon_s E_s = \eta \frac{d\varepsilon_d}{dt}$$

$$\dot{\varepsilon} = \frac{\dot{\sigma}}{E_s} + \frac{\sigma}{\eta}$$

$$\sigma(t) = E(t) * \frac{d\varepsilon(t)}{dt}$$

$$E(t) = E_s e^{-E_s t/\eta}$$

$$W = \frac{1}{2} E_s \left(\varepsilon - \varepsilon_d\right)^2 \qquad D = \frac{1}{2} \zeta \dot{\varepsilon}_d^2$$

$$\begin{bmatrix} E_s & -E_s \\ -E_s & E_s + \eta \frac{\partial}{\partial t} \end{bmatrix} \begin{Bmatrix} \varepsilon \\ \varepsilon_d \end{Bmatrix} = \begin{Bmatrix} \sigma \\ 0 \end{Bmatrix}$$

Viscoelastic material: $E(t) = E_e + \sum_{m=1}^{M} E_m e^{-t/\rho_m}$

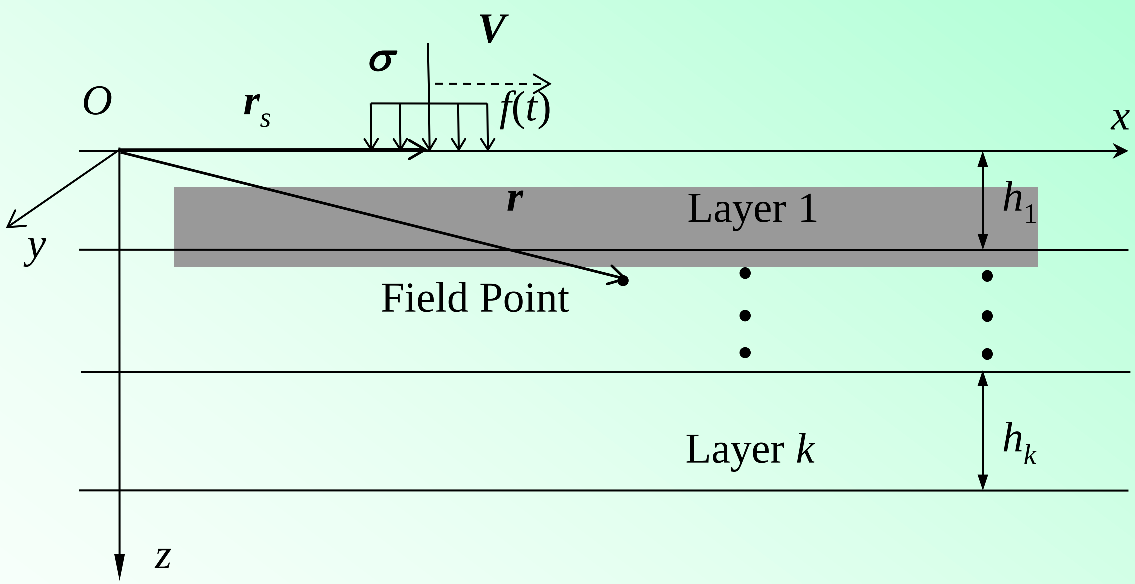
(homogeneous)

Eigenvalue problem $[M]$

Viscoelastic pavement:

(layered homogeneous)

Mathematic Model



Assumptions:
Homogeneous
Horizontally infinite
Isotropic
Full contact
Circular vertical load
(MEPDG 2004)

Pavement **Moving Dynamic** Response: ϕ

$$\phi(\boldsymbol{r},\boldsymbol{r}_s(t);t)=\int_{-\infty}^{t}\phi^{\delta}(x-x_{s0}-\boldsymbol{V}\tau, y-y_{s0}, z-z_{s0}; t-\tau)\,f(\tau)\,d\tau$$

Pavement **Impulse** Response: ϕ^{δ} $\quad \boldsymbol{L}[\phi^{\delta}(\boldsymbol{r}-\boldsymbol{r}_s;t-\tau)]=\boldsymbol{\sigma}\delta(t-\tau)$

Pavement **Stationary** Response: ϕ^{H} $\quad \boldsymbol{L}[\phi^{H}(\boldsymbol{r}-\boldsymbol{r}_s;t-\tau)]=\boldsymbol{\sigma}H(t-\tau)$

$$\phi^{\delta}(\eta,z;t)=\frac{d\phi^{H}}{dt}$$

Research Challenges

$$\phi(\mathbf{r},\mathbf{r}_s;t)=\int_{-\infty}^{t}\phi^{\delta}(x-x_{s0}-\mathbf{V}t,y-y_{s0},z-z_{s0};t-\tau)f(\tau)d\tau$$

Explicit Convolution Integral

$$L[\phi^{\delta}(\mathbf{r}-\mathbf{r}_s;t-\tau)]=\boldsymbol{\sigma}\delta(t-\tau)$$

Implicit Convolution Integral

Layered Viscoelastic Theory (LVET)

$$\sigma_{jl}=\lambda(t)*\frac{d\varepsilon_{kk}(t)}{dt}\delta_{jl}+2\mu(t)*\frac{d\varepsilon_{jl}(t)}{dt}$$

Layered Elastic Theory (LET)

$$\sigma_{jl}=\lambda\varepsilon_{kk}\delta_{jl}+2\mu\varepsilon_{jl}$$

$$\sigma_{jl}=\lambda(\eta;V;t)\varepsilon_{kk}(t)\delta_{jl}+2\mu(\eta;V;t)\varepsilon_{jl}(t)$$

Modified Elastic Model (LET)

Transformed domain: Correspondence Principle

$$\tilde{\sigma}_{jl}=(s\tilde{\lambda})\tilde{\varepsilon}_{kk}\delta_{jl}+2(s\tilde{\mu})\tilde{e}_{jl}$$

$$\boldsymbol{L}[\tilde{\phi}^{\delta}(\mathbf{r}-\mathbf{r}_s;s)]=\boldsymbol{\sigma}$$

$$\phi^{\delta}(t)=L^{-1}[\tilde{\phi}^{\delta}(s)]$$

Inverse Transform

$$\phi^{H}(t)=\sum_{k=1}^{K}\Gamma_k e^{-t/T_k}$$

Collocation Method

Time domain (FEM): **Convolution Integral**

$$(f*g)(t)=\int_{-\infty}^{t}f(t-\tau)g(\tau)d\tau$$

Convolution Integral

$$(f*g)(t)=\int_{-\infty}^{t} f(t-\tau)g(\tau)d\tau=\sum_{j=0}^{N_j-1} f(t-j\Delta\tau)g(\tau)\Delta\tau$$

Successive Algorithm
(Finite Element Method)

SUM(N)=1/9+2/9+3/9+4/9+5/9+…+N/9

SUM(N)=SUM(N-1)+N

…

N=4: 6/9+4/9=10/9

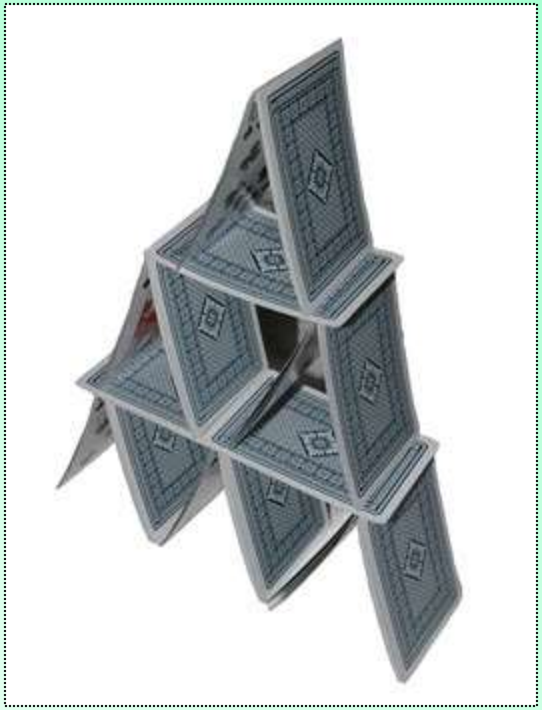
N=3: 3/9+3/9=6/9

N=2: 1/9+2/9=3/9

N=1: 1/9 =1/9

Deficiencies:

- ✓ Increased cost with increasing N
- ✓ Error accumulation (1/9=0.1, or 0.11 ?)
- ✓ Non-flexibility (N=100, 50, 200)



Objective

Method	Viscoelastic Treatment	Accuracy	Efficiency	Usage
FEM	Convolution Integral	X		X
LET	Modified Elastic Model		X	X
LVET	Convolution Integral	X		
	Inverse Laplace (Fourier) Transform	X		
	Collocation Method		X	
	? ?	X	X	X

Governing Equations

$$\phi(\mathbf{r},\mathbf{r}_s(t);t)=\int_{-\infty}^{t}\phi^{\delta}(\mathbf{r}-\mathbf{r}_s(\tau);t-\tau)\,f(\tau)\,d\tau$$

$$L[\phi^{\delta}(\mathbf{r}-\mathbf{r}_s;t-\tau)]=\boldsymbol{\sigma}\,\delta(t-\tau)$$

	Constitutive Equation	Strain-Displacement Equation	Equilibrium Equation	Boundary Condition
AC	$\sigma_{jl}=\lambda(t)*\frac{d\varepsilon_{kk}(t)}{dt}\delta_{jl}+2\mu(t)*\frac{d\varepsilon_{jl}(t)}{dt}$ $\tilde{\sigma}_{jl}=s\tilde{\lambda}(s)\tilde{\varepsilon}_{kk}(s)\delta_{jl}+s\tilde{\mu}(s)*\tilde{\varepsilon}_{jl}(s)$	$\varepsilon_{jl}=\frac{1}{2}(u_{l,j}+u_{j,l})$	$\sigma_{jl,l}=0$	$\sigma_{zz}(r,z=0;t)=-p_0\delta(t)H(d_0-r)$
BS	$\sigma_{jl}=\lambda\varepsilon_{kk}\delta_{jl}+2\mu\varepsilon_{jl}$			
SB	$\sigma_{jl}=\lambda\varepsilon_{kk}\delta_{jl}+2\mu\varepsilon_{jl}$			
SG	$\sigma_{jl}=\lambda\varepsilon_{kk}\delta_{jl}+2\mu\varepsilon_{jl}$			

Vector Functions

Cylindrical System

$$\boldsymbol{L} = \boldsymbol{i}_z S(r,\theta;\xi,m)$$

$$\boldsymbol{M} = grad(S) = \boldsymbol{i}_r \frac{\partial S}{\partial r} + \boldsymbol{i}_\theta \frac{\partial S}{r\partial \theta}$$

$$\boldsymbol{N} = curl(\boldsymbol{i}_z S) = \boldsymbol{i}_r \frac{\partial S}{r\partial \theta} - \boldsymbol{i}_\theta \frac{\partial S}{\partial r}$$

$$S(r,\theta;\xi,m) = \frac{1}{\sqrt{2\pi}} J_m(\xi r) e^{im\theta}$$

$$S^*(r,\theta;\xi,m) = \frac{1}{\sqrt{2\pi}} J_m(\xi r) e^{-im\theta}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial S}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 S}{\partial \theta^2} + \xi^2 S = 0$$

$$\boldsymbol{f}(r,\theta,z) = \sum_m \int_0^\infty \left[F_L \boldsymbol{L} + F_M \boldsymbol{M} + F_N \boldsymbol{N}\right] \xi d\xi$$

$$\boldsymbol{F}(L,M,N) = \int_0^{2\pi}\int_0^\infty \left(\boldsymbol{f}\cdot\boldsymbol{L}^* + \frac{1}{\xi^2}\boldsymbol{f}\cdot\boldsymbol{M}^* + \frac{1}{\xi^2}\boldsymbol{f}\cdot\boldsymbol{N}^*\right) r dr d\theta$$

Hankel Transform

Cartesian System

$$\boldsymbol{L} = \boldsymbol{i}_z S(x,y;\alpha,\beta)$$

$$\boldsymbol{M} = grad(S) = \boldsymbol{i}_x \frac{\partial S}{\partial x} + \boldsymbol{i}_y \frac{\partial S}{\partial y}$$

$$\boldsymbol{N} = curl(\boldsymbol{i}_z S) = \boldsymbol{i}_x \frac{\partial S}{\partial y} - \boldsymbol{i}_y \frac{\partial S}{\partial x}$$

$$S(x,y;\alpha,\beta) = \frac{1}{2\pi} e^{i(\alpha(x-x_{s0})+\beta(y-y_{s0}))}$$

$$S^*(x,y;\alpha,\beta) = \frac{1}{2\pi} e^{-i(\alpha(x-x_{s0})+\beta(y-y_{s0}))}$$

$$\frac{\partial^2 S}{\partial x^2} + \frac{\partial^2 S}{\partial y^2} + \xi^2 S = 0 \qquad \xi^2 = \alpha^2 + \beta^2$$

$$\boldsymbol{f}(x,y,z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[F_L \boldsymbol{L} + F_M \boldsymbol{M} + F_N \boldsymbol{N}\right] d\alpha d\beta$$

$$\boldsymbol{F}(L,M,N) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\boldsymbol{f}\cdot\boldsymbol{L}^* + \frac{1}{\xi^2}\boldsymbol{f}\cdot\boldsymbol{M}^* + \frac{1}{\xi^2}\boldsymbol{f}\cdot\boldsymbol{N}^*\right) dx dy$$

Fourier Transform

Pavement Primary Response

$$\boldsymbol{u}(x,y,z)=(u_x,u_y,u_z)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[U_L\boldsymbol{L}+U_M\boldsymbol{M}+U_N\boldsymbol{N}]d\alpha d\beta$$

$$\boldsymbol{t}(x,y,z)=(\sigma_{zx},\sigma_{zy},\sigma_{zz})=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[T_L\boldsymbol{L}+T_M\boldsymbol{M}+T_N\boldsymbol{N}]d\alpha d\beta$$

Integral Kernel

Pavement Primary Response:

$$\phi(x,y,z)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Phi(\xi,z)S(x,y;\alpha,\beta)d\alpha d\beta$$

$$S(x,y;\alpha,\beta)=\frac{1}{2\pi}e^{i(\alpha(x-x_{s0})+\beta(y-y_{s0}))}$$

ϕ	Φ
u_z	U_L
u_x	$U_M(-i\alpha)+U_N(-i\beta)$
u_y	$U_M(-i\beta)-U_N(-i\alpha)$
σ_z	T_L
σ_{zx}	$T_M(-i\alpha)+T_N(-i\beta)$
σ_{zy}	$T_M(-i\beta)-T_N(-i\alpha)$
…	(Table 3.1 Page 41)

Boundary Condition:

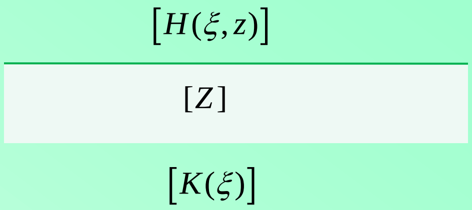
$$\sigma_{zz}(r,z=0;t)=-p_0\delta(t)H(d_0-r)$$
$$\sigma_{zr}=0$$
$$\sigma_{z\theta}=0$$

$$T_L(\xi,0;t)=\int\int\sigma_{zz}S^*dxdy=T_L(\xi)\delta(t)$$
$$T_M(\xi,0;t)=0$$
$$T_N(\xi,0;t)=0$$

Elastic Solution

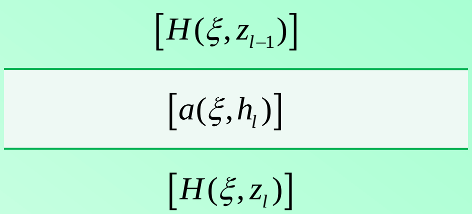
$$[H(\xi,z)] = [U_L \quad \xi U_M \quad T_L/\xi \quad T_M]^T = [Z(\xi,z)]_{4\times4}[K(\xi)]_{4\times1}$$

$$\left[H^N(\xi,z)\right]_{2\times1} = [U_N \quad T_N/\xi]^T = \left[Z^N(\xi,z)\right]_{2\times2}\left[K^N(\xi)\right]_{2\times1}$$



$$[H(\xi,z_{l-1})] = [a(\xi,h_l)][H(\xi,z_l)]$$

$$\left[H^N(\xi,z_{l-1})\right] = \left[a^N(\xi,h_l)\right]\left[H^N(\xi,z_l)\right]$$



$$[H(\xi,z)] = [A(\xi,z)][K(\xi)]$$

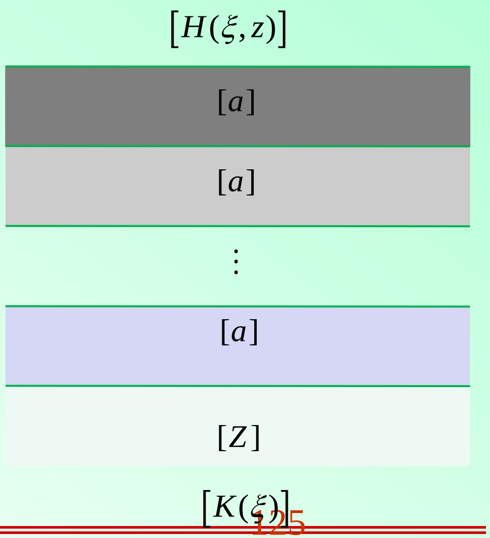
$$[A(\xi,z)] = [a(z_l - z)][a(h_{l+1})]\cdots[a(h_{k-1})][a(h_k)][Z(\xi,z_k)]$$

$$[K(\xi)]_{4\times1} = [[0]_{2\times1},[\kappa(\xi)]_{2\times1}]^T$$

$$\left[H^N(\xi,z)\right] = \left[A^N(\xi,z)\right]\left[K^N(\xi)\right]$$

$$\left[A^N(\xi,z)\right] = \left[a^N(z_l - z)\right]\left[a^N(h_{l+1})\right]\cdots\left[a^N(h_{k-1})\right]\left[a^N(h_k)\right]\left[Z^N(\xi,z_k)\right]$$

$$\left[K^N(\xi)\right]_{2\times1} = \left[0,\kappa^N(\xi)\right]^T \qquad T_N(\xi,0;t) = 0$$



Viscoelastic Solution

$$[H(\xi,z)]=[A(\xi,z)][K(\xi)]$$

$$\left[\tilde{H}(\xi,z;s)\right]=\left[\tilde{A}(\xi,z;s)\right]\left[\tilde{K}(\xi;s)\right]$$

$$[A(\xi,z)]=[a(z_l-z)][a(h_{l+1})]\cdots[a(h_{k-1})][a(h_k)][Z(\xi,z_k)]$$

$$\left[\tilde{A}(\xi,z;s)\right]=[\tilde{a}(z_1-z)][a(h_2)]\cdots[a(h_{k-1})][a(h_k)][Z(\xi,z_k)]$$

AC

$$[a]=\begin{bmatrix}[a]_{11} & \vdots & \frac{1}{E}[a]_{12}\\ \cdots & \cdot & \cdots \\ E[a]_{21} & \vdots & [a]_{22}\end{bmatrix}$$

$$[\tilde{a}]=\begin{bmatrix}[a]_{11} & \vdots & \frac{1}{s\tilde{E}}[a]_{12}\\ \cdots & \cdot & \cdots \\ s\tilde{E}[a]_{21} & \vdots & [a]_{22}\end{bmatrix}$$

$$\frac{1}{s\tilde{E}(s)}=s\tilde{D}(s)$$

$$[\tilde{a}]=\begin{bmatrix}[a]_{11} & \vdots & s\tilde{D}(s)[a]_{12}\\ \cdots & \cdot & \cdots \\ s\tilde{E}[a]_{21} & \vdots & [a]_{22}\end{bmatrix}$$

Layer 2: $[a]$ | $[a]$ | $[a]$

$\vdots$ | $\vdots$ | $\vdots$

$[a]$ | $[a]$ | $[a]$

SG: $[Z]$ | $[Z]$ | $[Z]$

Viscoelastic Solution

$$\left[\tilde{H}(\xi,z;s)\right]=\left(\left[A_0(\xi,z)\right]+\left[A_E(\xi,z)\right]\sum_{m=1}^{M}\frac{-E_m}{s\rho_m+1}+\left[A_D(\xi,z)\right]\sum_{m=1}^{M}\frac{D_m}{s\tau_m+1}\right)\left[\tilde{K}(\xi;s)\right]$$

$$\left[H(\xi,z;t)\right]=\left[A_0(\xi,z)\right]\left[K(\xi;t)\right]+\left[A_E(\xi,z)\right]\sum_{m=1}^{M}\frac{-E_m}{\rho_m}e^{-t/\rho_i}*\left[K(\xi;t)\right]+\left[A_D(\xi,z)\right]\sum_{m=1}^{M}\frac{D_m}{\tau_m}e^{-t/\tau_m}*\left[K(\xi;t)\right]$$

$$\left[K(\xi)\right]_{4\times1}=\left[[0]_{2\times1},[\kappa(\xi)]_{2\times1}\right]^T$$

$$\left[U(\xi,z;t)\right]=\begin{Bmatrix}U_L\\ \xi U_M\end{Bmatrix}=\left[A_0(\xi,z)\right]_{12}\left[\kappa(\xi;t)\right]+\left[A_D(\xi,z)\right]_{12}\sum_{m=1}^{M}\frac{D_m}{\tau_m}e^{-t/\tau_m}*\left[\kappa(\xi;t)\right]$$

$$\left[T(\xi,z;t)\right]=\begin{Bmatrix}T_L/\xi\\ T_M\end{Bmatrix}=\left[A_0(\xi,z)\right]_{22}\left[\kappa(\xi;t)\right]+\left[A_E(\xi,z)\right]_{22}\sum_{m=1}^{M}\frac{-E_m}{\rho_m}e^{-t/\rho_m}*\left[\kappa(\xi;t)\right]$$

$$T_L(\xi,0;t)=\iint\sigma_{zz}S^*dxdy=T_L(\xi)\delta(t)$$

$$T_M(\xi,0;t)=0$$

Pavement Impulse Response

$$[T(\xi,0;t)]=[A_0(\xi,0)]_{22}[\kappa(\xi;t)]+[A_E(\xi,0)]_{22}\sum_{m=1}^{M}\frac{-E_m}{\rho_m}e^{-t/\rho_m}*[\kappa(\xi;t)]$$

$$[\kappa(\xi;t)]=[g(\xi)]\delta(t)-[C(\xi)]\sum_{m=1}^{M}l_m e^{-t/\rho_m}*[\kappa(\xi;t)]$$

Volterra system of equations of second kind

$$[Q(\xi)]=\begin{bmatrix} [C(\xi)]l_1+\frac{1}{\rho_1}I_0 & [C(\xi)]l_2 & \mathrm{L} & [C(\xi)]l_m & \mathrm{L} & [C(\xi)]l_M \\ [C(\xi)]l_1 & [C(\xi)]l_2+\frac{1}{\rho_2}I_0 & \mathrm{L} & [C(\xi)]l_m & \mathrm{L} & [C(\xi)]l_M \\ \mathrm{M} & \mathrm{M} & \mathrm{M} & \mathrm{M} & \mathrm{M} & \mathrm{M} \\ [C(\xi)]l_1 & [C(\xi)]l_2 & \mathrm{L} & [C(\xi)]l_m+\frac{1}{\rho_m}I_0 & \mathrm{L} & [C(\xi)]l_M \\ \mathrm{M} & \mathrm{M} & \mathrm{M} & \mathrm{M} & \mathrm{M} & \mathrm{M} \\ [C(\xi)]l_1 & [C(\xi)]l_2 & \mathrm{L} & [C(\xi)]l_m & \mathrm{L} & [C(\xi)]l_M+\frac{1}{\rho_M}I_0 \end{bmatrix}$$

Eigenvalue problem $[Q]$

$$[X]\langle\omega_j\rangle[X]^{-1}=-[Q]$$

$$[\kappa(\xi;t)]_{2\times1}=[\kappa_0(\xi;t)]_{2\times1}\delta(t)+\sum_{j=1}^{2M}[\kappa_j(\xi;t)]_{2\times1}e^{\omega_j(\xi)t}H(t)$$

$$\Phi^{\delta}(\alpha,\beta,z;t)=\Phi_0(\alpha,\beta,z)\delta(t)+\sum_{n}\Phi_n(\alpha,\beta,z)e^{\omega_n(\xi)t}H(t)$$

Semianalytical Solution

$$\phi^{\delta}(\mathbf{r},\mathbf{r}_s(t);t)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Phi^{\delta}(\alpha,\beta,z;t)S(x(t),y;\alpha,\beta)d\alpha d\beta$$

Pavement Impulse Response

$$\Phi^{\delta}(\alpha,\beta,z;t)=\Phi_0(\alpha,\beta,z)\delta(t)+\sum_n \Phi_n(\alpha,\beta,z)e^{\omega_n(\xi)t}H(t)$$

$$\phi^{\delta}(\mathbf{r},\mathbf{r}_s(t);t)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Phi^{\delta}(\alpha,\beta,z;t)S(x(t),y;\alpha,\beta)d\alpha d\beta$$

ϕ^{δ}	Φ^{δ}
u_z^{δ}	U_L^{δ}
u_x^{δ}	$U_M^{\delta}(-i\alpha) + U_N^{\delta}(-i\beta)$
u_y^{δ}	$U_M^{\delta}(-i\beta) - U_N^{\delta}(-i\alpha)$
…	

$$\Phi^{\delta}=\sum_q [\Omega^{(q)}(\xi)]^{\delta}\alpha^{k_\alpha^{(q)}}\beta^{k_\beta^{(q)}}$$

$$[\Omega^{(q)}(\xi,z;t)]^{\delta}=\Omega_0^{(q)}(\xi,z)\delta(t)+\sum_{n=1}\Omega_n^{(q)}(\xi,z)e^{\omega_n^{(q)}(t)}H(t)$$

$$\phi^{\delta}(\mathbf{r},\mathbf{r}_s(t);t)=\sum_{q=1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[\Omega^{(q)}(\xi,z;t)]^{\delta}\alpha^{k_\alpha^{(q)}}\beta^{k_\beta^{(q)}}S(x(t),y;\alpha,\beta)d\alpha d\beta$$

ϕ^{δ}	q	$[\Omega^{(q)}(\xi)]^{\delta}$	$k_\alpha^{(q)}$	$k_\beta^{(q)}$
u_z^{δ}	1	U_L^{δ}	0	0
u_x^{δ}	2	$U_M^{\delta}(-i)$	1	0
		$U_N^{\delta}(-i)$	0	1
u_y^{δ}	2	$U_M^{\delta}(-i)$	0	1
		$- U_N^{\delta}(-i)$	1	0
…	…			

Pavement Stationary Response

$$\phi^{\delta}(\mathbf{r},\mathbf{r}_s(t);t)=\sum_{q=1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[\Omega^{(q)}(\xi,z;t)]^{\delta}\alpha^{k_\alpha^{(q)}}\beta^{k_\beta^{(q)}}S(x(t),y;\alpha,\beta)d\alpha d\beta$$

$$\phi^{H}(\mathbf{r},\mathbf{r}_s;t)=\sum_{q}\int_0^t\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[\Omega^{(q)}(\xi,z;t-\tau)]^{\delta}\alpha^{k_\alpha^{(q)}}\beta^{k_\beta^{(q)}}S(x,y;\alpha,\beta)d\alpha d\beta d\tau$$

$$\phi^{H}(\mathbf{r},\mathbf{r}_s;t)=\sum_{q}\int_0^{\infty}\left[\int_0^t[\Omega^{(q)}(\xi,z;t-\tau)]^{\delta}d\tau\right]\xi^{k_\alpha^{(q)}+k_\beta^{(q)}+1}C^{k_\alpha^{(q)},k_\beta^{(q)}}(r,\theta;\xi)d\xi$$

Semianalytical Solution

$$[\Omega^{(q)}(\xi,z;t)]^{\delta}=\Omega_0^{(q)}(\xi,z)\delta(t)+\sum_{n=1}\Omega_n^{(q)}(\xi,z)e^{\omega_n^{(q)}(t)}H(t)$$

$$\int_0^t[\Omega^{(q)}(\xi,z;t-\tau)]^{\delta}d\tau=\Omega_0^{(q)}+\sum_n\frac{\Omega_n^{(q)}}{\omega_n^{(q)}}(e^{\omega_n^{(q)}t}-1)H(t)$$

ϕ	q	$[\Omega^{(q)}(\xi,z;t-\tau)]^{\delta}$	$C^{k_\alpha^{(q)},k_\beta^{(q)}}(r,\theta;\xi)$
u_z	1	U_L^{δ}	$C^{0,0}$
u_x	2	U_M^{δ}(-i)	$C^{1,0}$
		U_N^{δ}(-i)	$C^{0,1}$
u_y	2	U_M^{δ}(-i)	$C^{0,1}$
		- U_N^{δ}(-i)	$C^{1,0}$
…	…		

$$C^{k_\alpha^{(q)},k_\beta^{(q)}}(r,\theta;\xi)=\frac{1}{2\pi}\int_0^{2\pi}(\cos\varphi)^{k_\alpha^{(q)}}(\sin\varphi)^{k_\beta^{(q)}}e^{i\xi r\cos(\theta-\varphi)}d\varphi$$

$C^{0,0}$	1	$CC^{0,0}$	$J_0(\xi r)$	$C^{1,1}$	1	$CC^{1,1}$	$-\frac{1}{2}J_2(\xi r)\sin(2\theta)$
	i	$CS^{0,0}$	0		i	$CS^{1,1}$	0
$C^{1,0}$	1	$CC^{1,0}$	0	$C^{2,0}$	1	$CC^{2,0}$	$\frac{1}{2}J_0(\xi r)-\frac{1}{2}J_2(\xi r)\cos(2\theta)$
	i	$CS^{1,0}$	$J_1(\xi r)\sin\theta$		i	$CS^{2,0}$	0
$C^{0,1}$	1	$CC^{0,1}$	0	$C^{0,2}$	1	$CC^{0,2}$	$\frac{1}{2}J_0(\xi r)+\frac{1}{2}J_2(\xi r)\cos(2\theta)$
	i	$CS^{0,1}$	$J_1(\xi r)\cos\theta$		i	$CS^{0,2}$	0

Numerical Verification

Xu (2004), *Ph.D. Dissertation*, North Carolina Univ.: FEM

Layer	Thickness (in)	Young's Modulus (psi)	Poisson's Ratio
AC	8.0	Viscoelastic	0.35
Subbase	8.0	30,000	0.3
Subgrade	Infinite	5,000	0.3

i	$E(t)$		$D(t)$	
	E_i (psi)	❐$_i$ (s)	D_i (psi^{-1})	♦$_i$ (s)
	E_e=1.25×10^4	-	D_0=7.92451×10^{-7}	-
1	7.353×10^5	0.008441	8.72639×10^{-7}	0.0189201
2	3.862×10^5	0.1319	3.56881×10^{-9}	0.452776
3	1.075×10^5	1.968	1.80369×10^{-5}	8.61009
4	2.036×10^4	39.25	5.67292×10^{-5}	117.703

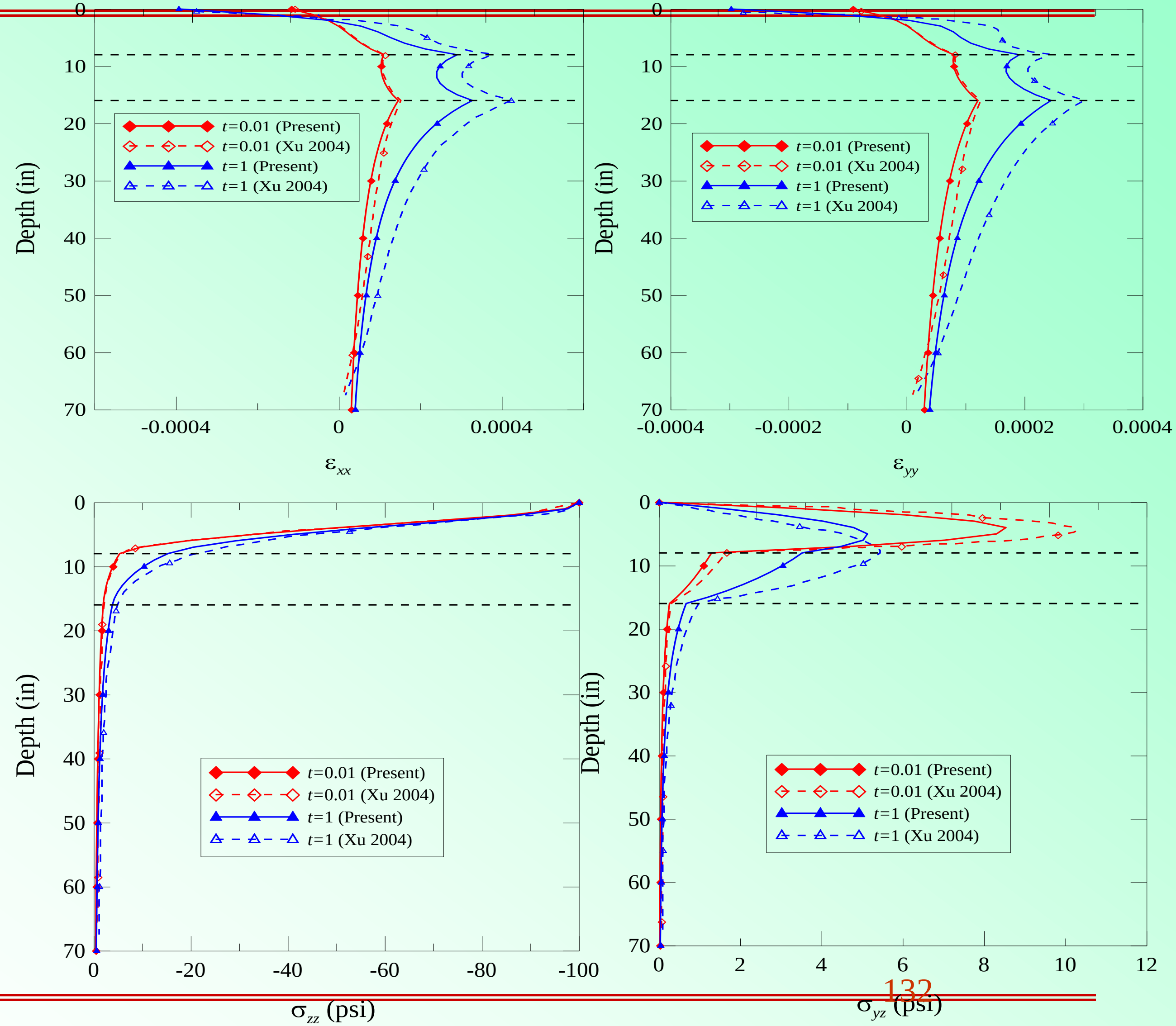
Numerical Verification

FEM Deficiencies:

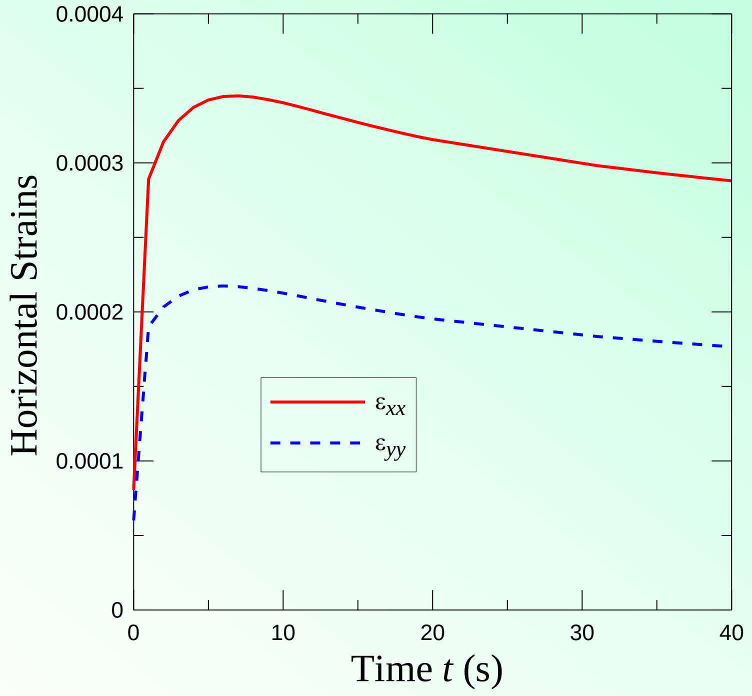
✓ Increased cost with increasing time steps

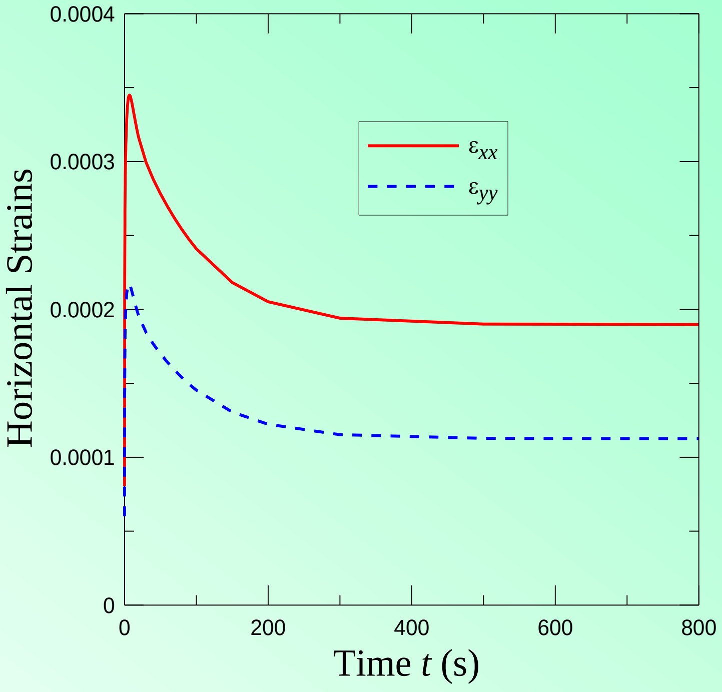
✓ **Error accumulation**

✓ Non-flexibility



Long-term Prediction





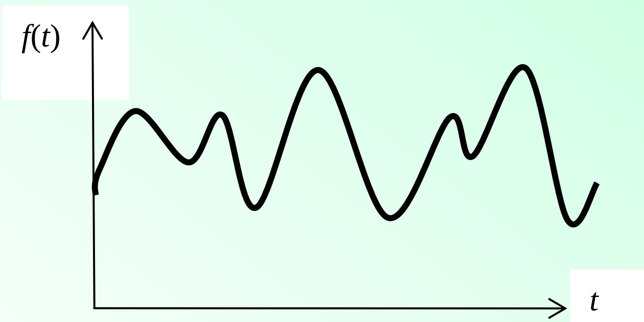
FEM Deficiencies:

- ✓ **Increased cost with increasing time steps**
- ✓ Error accumulation
- ✓ Non-flexibility

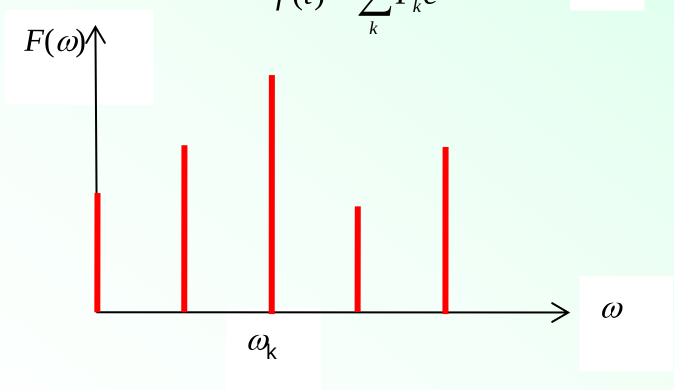
Pavement Moving Dynamic Response

$$\phi^{\delta}(\mathbf{r},\mathbf{r}_s(t);t)=\sum_{q=1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[\Omega^{(q)}(\xi,z;t)]^{\delta}\alpha^{k_\alpha^{(q)}}\beta^{k_\beta^{(q)}}S(x(t),y;\alpha,\beta)\,d\alpha\, d\beta$$

$$\phi(\mathbf{r},\mathbf{r}_s(t);t)=\int_{-\infty}^{t}\phi^{\delta}(x-x_{s0}-\mathbf{V}\tau,\ y-y_{s0},\ z-z_{s0};\ t-\tau)\,f(\tau)\,d\tau$$



$$f(t)=\sum_k F_k e^{i\omega_k t}$$



$$\phi(\mathbf{r},\mathbf{r}_s(t);t)=\sum_k\sum_q F_k\int_0^{\infty}\left\{\begin{array}{l}\Omega_0^{(q)}C^{k_{\alpha j},k_{\beta j}}(r(t),\theta(t);\xi;\rho=0)e^{i\omega_k t}+\\ \sum_n\dfrac{\Omega_n^{(q)}}{-\omega_n^{(q)}+i\omega_k}\left(\begin{array}{l}C^{k_\alpha^{(q)},k_\beta^{(q)}}(r(t),\theta(t);\xi;\rho)e^{i\omega_k t}\\ -C^{k_\alpha^{(q)},k_\beta^{(q)}}(r(0),\theta(0);\xi;\rho)e^{\omega_n^{(q)}t}\end{array}\right)\end{array}\right\}\xi^{k_\alpha^{(q)}+k_\beta^{(q)}+1}d\xi$$

$$\rho_{nk}^{(q)}=\frac{\xi\mathbf{V}}{\omega_n^{(q)}-i\omega_k}$$

ϕ	q	$[\Omega^{(q)}(\xi,z;t-\tau)]^{\delta}$	$C^{k_\alpha^{(q)},k_\beta^{(q)}}(r,\theta;\xi;\rho)$
u_z	1	$\mathbf{U}_L$	$C^{0,0}$
u_x	2	$\mathbf{U}_M(-i)$	$C^{1,0}$
		$\mathbf{U}_N(-i)$	$C^{0,1}$
u_y	2	$\mathbf{U}_M(-i)$	$C^{0,1}$
		$-\mathbf{U}_N(-i)$	$C^{1,0}$
…	…		

Pavement Moving Dynamic Response

$$\phi(\mathbf{r},\mathbf{r}_s(t);t)=\sum_k\sum_q F_k\int_0^\infty \left\{ \begin{array}{l} \Omega_0^{(q)} C^{k_{\alpha j},k_{\beta j}}(r(t),\theta(t);\xi;\rho=0)e^{i\omega_k t} + \\ \sum_n \frac{\Omega_n^{(q)}}{-\omega_n^{(q)}+i\omega_k}\left(\begin{array}{l} C^{k_\alpha^{(q)},k_\beta^{(q)}}(r(t),\theta(t);\xi;\rho)e^{i\omega_k t} \\ -C^{k_\alpha^{(q)},k_\beta^{(q)}}(r(0),\theta(0);\xi;\rho)e^{\omega_n^{(q)}t}\end{array}\right)\end{array}\right\}\xi^{k_\alpha^{(q)}+k_\beta^{(q)}+1}d\xi$$

$$C^{k_\alpha^{(q)},k_\beta^{(q)}}(r,\theta;\xi;\rho)=\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{i\xi r\cos(\varphi-\theta)}}{1+i\rho\cos\varphi}(\cos\varphi)^{k_\alpha^{(q)}}(\sin\varphi)^{k_\beta^{(q)}}d\varphi$$

$C^{0,0}$	1	$CC^{0,0}$	$J_0(\xi r)NC_0(\rho)+2\sum_{m=1}(-)^m J_{2m}(\xi r)NC_{2m}(\rho)\cos(2m\theta)$
	ρ	$CS^{1,0}$	$\sum_{m=0}(-)^m J_{2m+1}(\xi r)\left(NC_{2m}(\rho)+NC_{2m+2}(\rho)\right)\cos((2m+1)\theta)$
…	…	…	

$$C^{k_\alpha^{(q)},k_\beta^{(q)}}(r,\theta;\xi)=\frac{1}{2\pi}\int_0^{2\pi}(\cos\varphi)^{k_\alpha^{(q)}}(\sin\varphi)^{k_\beta^{(q)}}e^{i\xi r\cos(\theta-\varphi)}d\varphi$$

$C^{0,0}$	1	$CC^{0,0}$	$J_0(\xi r)$	$C^{1,1}$	1	$CC^{1,1}$	$-\frac{1}{2}J_2(\xi r)\sin(2\theta)$
	i	$CS^{0,0}$	0		i	$CS^{1,1}$	0
…	…	…	…	…			

$$NC_{2m}(\rho)=\frac{(-1)^m\zeta^{2m}}{\sqrt{1+\rho^2}}=\frac{(-1)^m}{\sqrt{1+\rho^2}}\left(\frac{\sqrt{1+\rho^2}-1}{\rho}\right)^{2m}$$

$$\rho_{nk}^{(q)}=\frac{\xi\mathbf{V}}{\omega_n^{(q)}-i\omega_k}$$

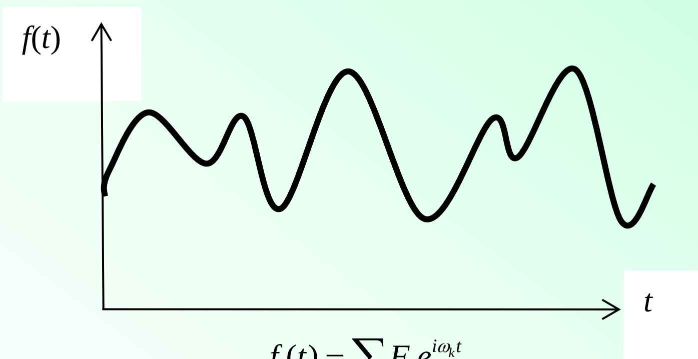
$$C^{k_\alpha^{(q)},k_\beta^{(q)}}(r,\theta;\xi;\rho)=\sum_{m=0}J_m(\xi r)\chi_m^{k_\alpha^{(q)},k_\beta^{(q)}}(\theta,\rho)$$

$$\phi(\mathbf{r},\mathbf{r}_s(t);t)=\sum_k\sum_q\sum_m F_k\int_0^\infty \left\{ \begin{array}{l} \Omega_0^{(q)} J_m(\xi r(t))\chi_m^{k_\alpha^{(q)},k_\beta^{(q)}}(\theta(t);\rho=0)e^{i\omega_k t} + \\ \sum_n \frac{\Omega_n^{(q)}}{-\omega_n^{(q)}+i\omega_k}\left(\begin{array}{l} J_m(\xi r(t))\chi_m^{k_\alpha^{(q)},k_\beta^{(q)}}(\theta(t);\rho)e^{i\omega_k t} \\ -J_m(\xi r(0))\chi_m^{k_\alpha^{(q)},k_\beta^{(q)}}(\theta(0);\rho)e^{\omega_n^{(q)}t}\end{array}\right)\end{array}\right\}\xi^{k_\alpha^{(q)}+k_\beta^{(q)}+1}d\xi$$

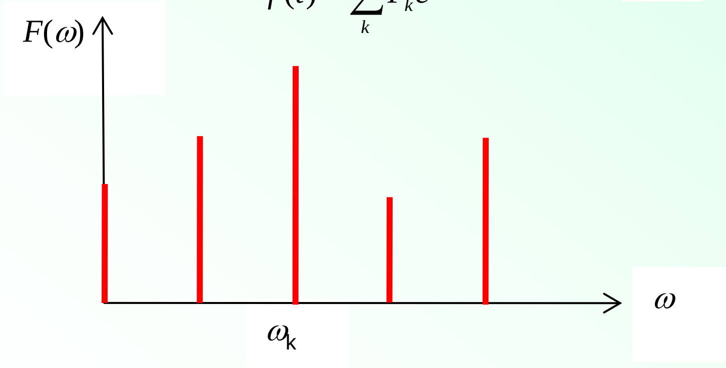
Semianalytical Solution

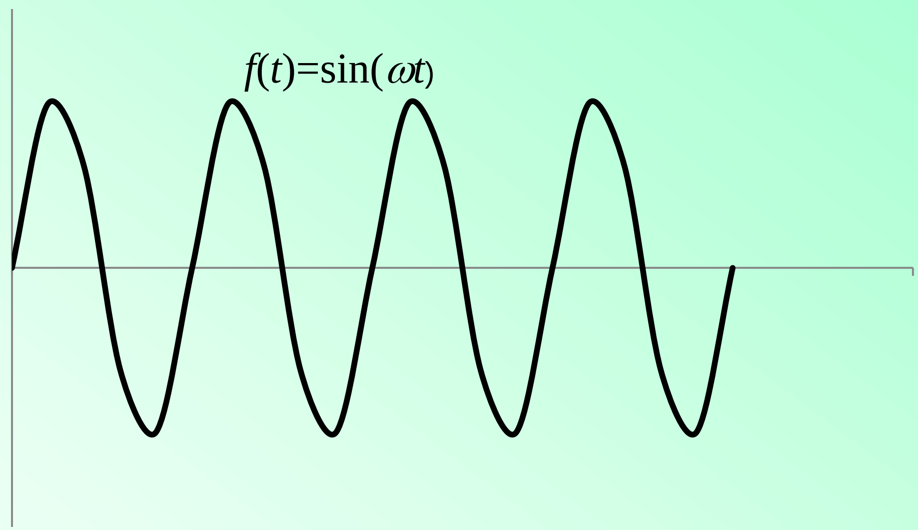
Index *k*

$$\phi(\mathbf{r},\mathbf{r}_s(t);t)=\sum_k\sum_q\sum_m F_k\int_0^\infty\left\{\begin{array}{l}\Omega_0^{(q)}J_m(\xi r(t))\chi_m^{k_\alpha^{(q)},k_\beta^{(q)}}(\theta(t);\rho=0)e^{i\omega_k t}+\\ \displaystyle\sum_n\frac{\Omega_n^{(q)}}{-\omega_n^{(q)}+i\omega_k}\left(\begin{array}{l}J_m(\xi r(t))\chi_m^{k_\alpha^{(q)},k_\beta^{(q)}}(\theta(t);\rho)e^{i\omega_k t}\\ -J_m(\xi r(0))\chi_m^{k_\alpha^{(q)},k_\beta^{(q)}}(\theta(0);\rho)e^{\omega_n^{(q)}t}\end{array}\right)\end{array}\right\}\xi^{k_\alpha^{(q)}+k_\beta^{(q)}+1}d\xi$$



$$f(t)=\sum_k F_k e^{i\omega_k t}$$





k: dynamic load

Index q

$$\phi(\mathbf{r},\mathbf{r}_s(t);t)=\sum_k\sum_q\sum_m F_k\int_0^\infty\left\{\begin{array}{l}\Omega_0^{(q)}J_m(\xi r(t))\chi_m^{k_\alpha^{(q)},k_\beta^{(q)}}(\theta(t);\rho=0)e^{i\omega_k t}+\\ \sum_n\frac{\Omega_n^{(q)}}{-\omega_n^{(q)}+i\omega_k}\left(\begin{array}{l}J_m(\xi r(t))\chi_m^{k_\alpha^{(q)},k_\beta^{(q)}}(\theta(t);\rho)e^{i\omega_k t}\\ -J_m(\xi r(0))\chi_m^{k_\alpha^{(q)},k_\beta^{(q)}}(\theta(0);\rho)e^{\omega_n^{(q)}t}\end{array}\right)\end{array}\right\}\xi^{k_\alpha^{(q)}+k_\beta^{(q)}+1}d\xi$$

ϕ^δ	q	$[\Omega^{(q)}(\xi)]^\delta$	$k_\alpha^{(q)}$	$k_\beta^{(q)}$
u_z^δ	1	U_L^δ	0	0
u_x^δ	2	U_M^δ (-i)	1	0
		U_N^δ (-i)	0	1
u_y^δ	2	U_M^δ (-i)	0	1
		- U_N^δ (-i)	1	0
…	…			

q: response type
boundary condition

Index *m*

$$\phi(\mathbf{r},\mathbf{r}_s(t);t)=\sum_k\sum_q\sum_m F_k\int_0^\infty\left\{\begin{array}{l}\Omega_0^{(q)}J_m(\xi r(t))\chi_m^{k_\alpha^{(q)},k_\beta^{(q)}}(\theta(t);\rho=0)e^{i\omega_k t}+\\ \sum_n\frac{\Omega_n^{(q)}}{-\omega_n^{(q)}+i\omega_k}\left(\begin{array}{l}J_m(\xi r(t))\chi_m^{k_\alpha^{(q)},k_\beta^{(q)}}(\theta(t);\rho)e^{i\omega_k t}\\ -J_m(\xi r(0))\chi_m^{k_\alpha^{(q)},k_\beta^{(q)}}(\theta(0);\rho)e^{\omega_n^{(q)}t}\end{array}\right)\end{array}\right\}\xi^{k_\alpha^{(q)}+k_\beta^{(q)}+1}d\xi$$

$$C^{k_\alpha^{(q)},k_\beta^{(q)}}(r,\theta;\xi;\rho)=\sum_{m=0}J_m(\xi r)\chi_m^{k_\alpha^{(q)},k_\beta^{(q)}}(\theta,\rho)$$

$$\rho_{nk}^{(q)}=\frac{\xi V}{\omega_n^{(q)}-i\omega_k}$$

$$C^{k_\alpha^{(q)},k_\beta^{(q)}}(r,\theta;\xi;\rho)=\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{i\xi r\cos(\varphi-\theta)}}{1+i\rho\cos\varphi}(\cos\varphi)^{k_\alpha^{(q)}}(\sin\varphi)^{k_\beta^{(q)}}d\varphi$$

$C^{0,0}$	1	$CC^{0,0}$	$J_0(\xi r)NC_0(\rho)+2\sum_{m=1}(-)^m J_{2m}(\xi r)NC_{2m}(\rho)\cos(2m\theta)$
	ρ	$CS^{1,0}$	$\sum_{m=0}(-)^m J_{2m+1}(\xi r)\left(NC_{2m}(\rho)+NC_{2m+2}(\rho)\right)\cos((2m+1)\theta)$

$$C^{k_\alpha^{(q)},k_\beta^{(q)}}(r,\theta;\xi)=\frac{1}{2\pi}\int_0^{2\pi}(\cos\varphi)^{k_\alpha^{(q)}}(\sin\varphi)^{k_\beta^{(q)}}e^{i\xi r\cos(\theta-\varphi)}d\varphi$$

$C^{0,0}$	1	$CC^{0,0}$	$J_0(\xi r)$	$C^{1,1}$	1	$CC^{1,1}$	$-\frac{1}{2}J_2(\xi r)\sin(2\theta)$
	i	$CS^{0,0}$	0		i	$CS^{1,1}$	0
…	…	…	…	…			

m: moving or stationary load

Index *n*

$$\phi(\mathbf{r},\mathbf{r}_s(t);t)=\sum_k\sum_q\sum_m F_k\int_0^\infty \left\{ \begin{array}{l} \Omega_0^{(q)} J_m(\xi r(t))\chi_m^{k_\alpha^{(q)},k_\beta^{(q)}}(\theta(t);\rho=0)e^{i\omega_k t}+ \\ \sum_n \dfrac{\Omega_n^{(q)}}{-\omega_n^{(q)}+i\omega_k}\left(\begin{array}{l} J_m(\xi r(t))\chi_m^{k_\alpha^{(q)},k_\beta^{(q)}}(\theta(t);\rho)e^{i\omega_k t} \\ -J_m(\xi r(0))\chi_m^{k_\alpha^{(q)},k_\beta^{(q)}}(\theta(0);\rho)e^{\omega_n^{(q)}t} \end{array} \right) \end{array} \right\} \xi^{k_\alpha^{(q)}+k_\beta^{(q)}+1}d\xi$$

Eigenvalue problem $[Q]$

$$[Q(\xi)]=\begin{bmatrix} [C(\xi)]l_1+\dfrac{1}{\rho_1}I_0 & [C(\xi)]l_2 & \cdots & [C(\xi)]l_m & \cdots & [C(\xi)]l_M \\ [C(\xi)]l_1 & [C(\xi)]l_2+\dfrac{1}{\rho_2}I_0 & \cdots & [C(\xi)]l_m & \cdots & [C(\xi)]l_M \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ [C(\xi)]l_1 & [C(\xi)]l_2 & \cdots & [C(\xi)]l_m+\dfrac{1}{\rho_m}I_0 & \cdots & [C(\xi)]l_M \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ [C(\xi)]l_1 & [C(\xi)]l_2 & \cdots & [C(\xi)]l_m & \cdots & [C(\xi)]l_M+\dfrac{1}{\rho_M}I_0 \end{bmatrix}$$

$$E(t)=E_e+\sum_{m=1}^{M}E_m e^{-t/\rho_m} \qquad D(t)=D_0+\sum_{m=1}^{M}D_m(1-e^{-t/\tau_m})$$

n: viscoelastic properties

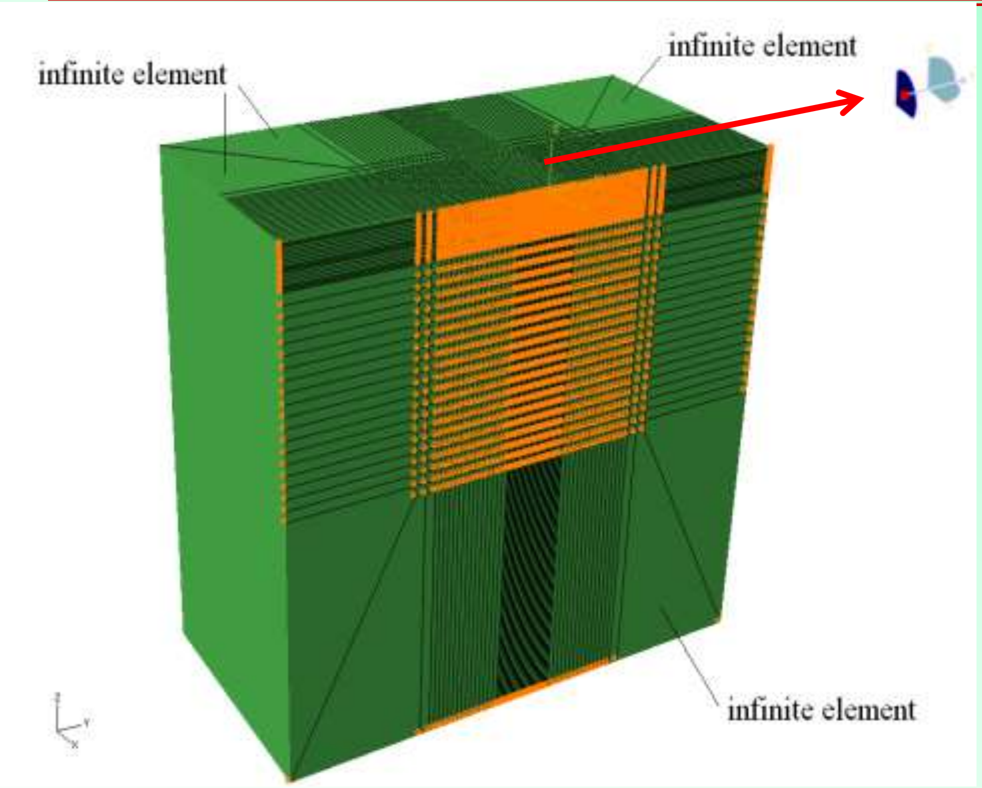
Numerical Verification

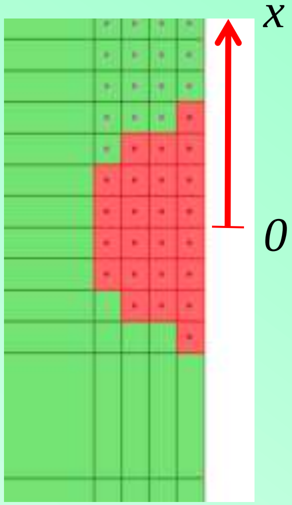
Layer	Thickness (in)	Young's Modulus (psi)	Poisson's Ratio
AC	16.0	Viscoelastic	0.35
Subbase	12.0	37,990	0.4
Subgrade	Infinite	7,500	0.45

Al-Qadi et al. (2008), *TRR 2087*

i	$E(t)$		$D(t)$	
	E_i (psi)	❐$_i$ (s)	D_i (psi^{-1})	♦$_i$ (s)
	E_e=5.943E+03	-	D_0=3.668E-7	-
1	1.232E+06	1.130E-04	2.799E-07	2.027E-04
2	7.578E+05	3.140E-03	3.429E-07	5.487E-03
3	4.034E+05	1.300E-02	1.440E-06	3.050E-02
4	2.944E+05	1.840E-01	8.577E-06	1.187E+00
5	2.034E+04	2.290E+00	2.968E-05	6.303E+00
6	1.189E+04	2.570E+01	1.276E-04	9.032E+01

Abaqus Model

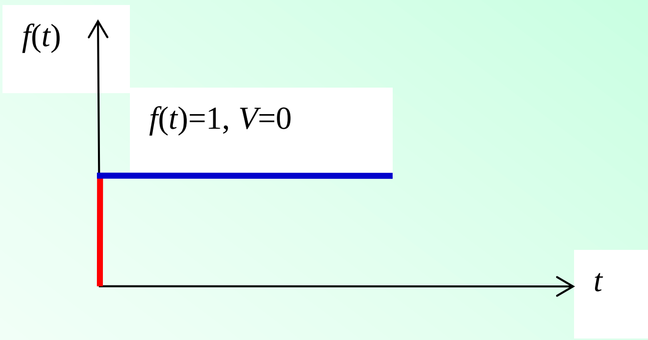


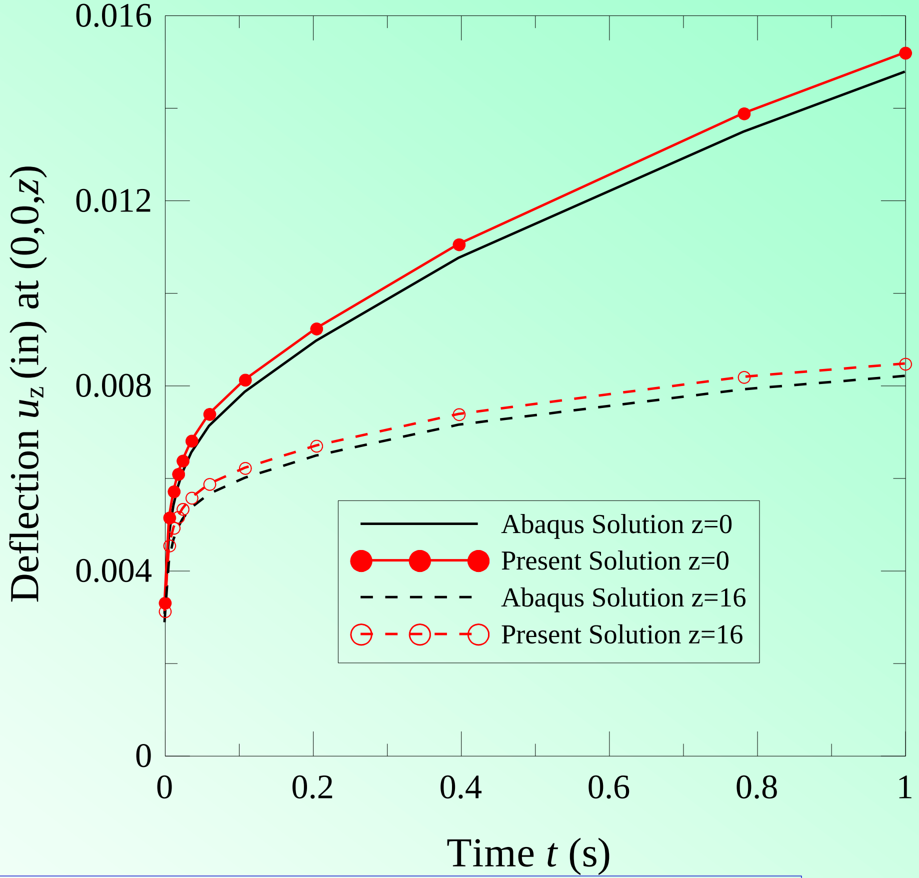


d_0=3.909

Total Number of Elements: 92,772
Total Number of Nodes: 99,522
Element type: C3D8R,CIN3D8

Verification-Stationary Load

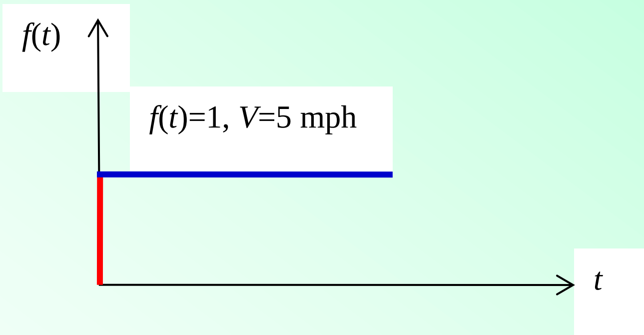




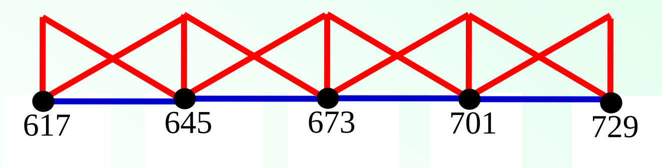
Abaqus CPU time: 18231/14 s
Semianalytical Solution: <1 s

DELL OptiPlex 755	DELL Precision 360
Intel Core 2 Quad CPU/Q6600 2.40GHz	Intel Pentium 4 CPU/3.40 GHz
RAM 3.25 GB	RAM 2.00 GB

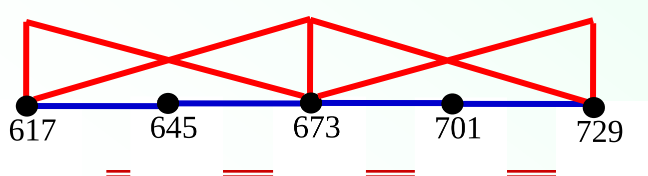
Verification-Moving Load

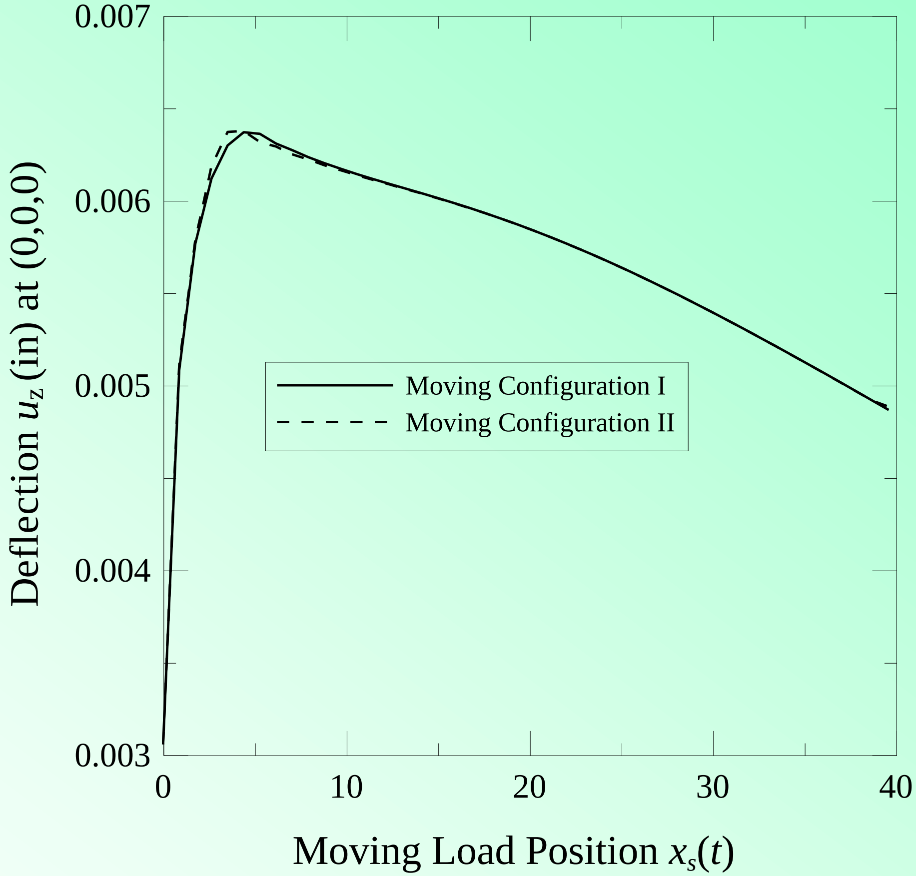


Moving Configuration I

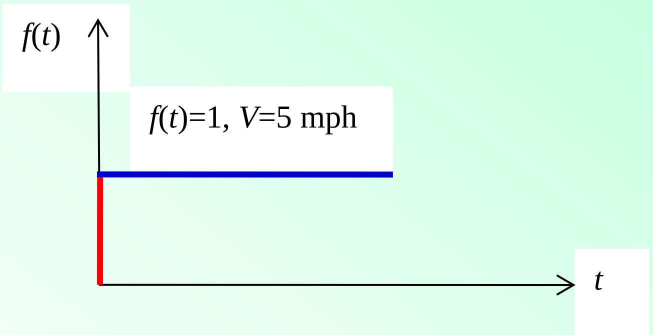


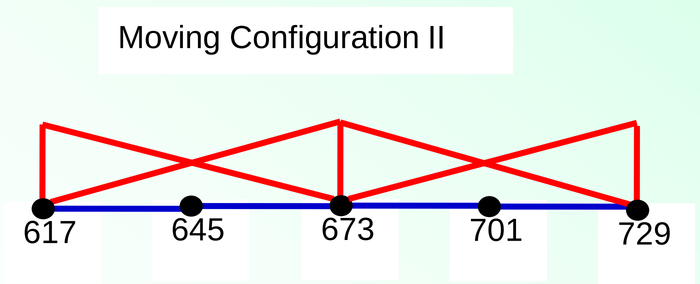
Moving Configuration II

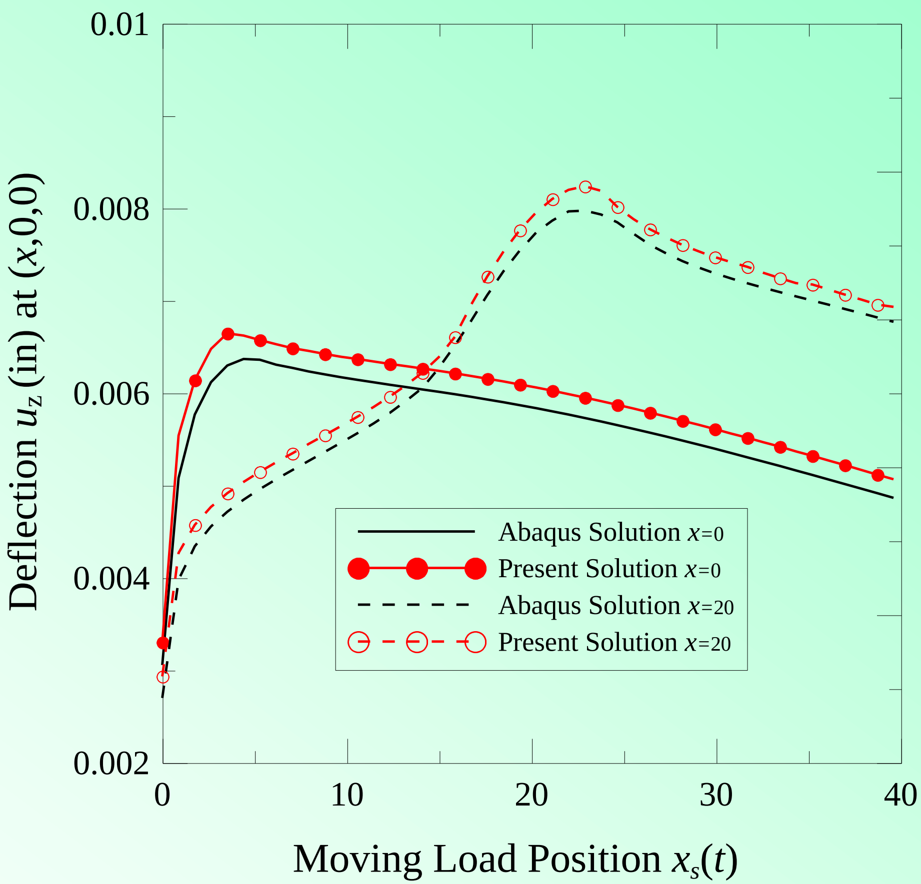




Verification-Moving Load



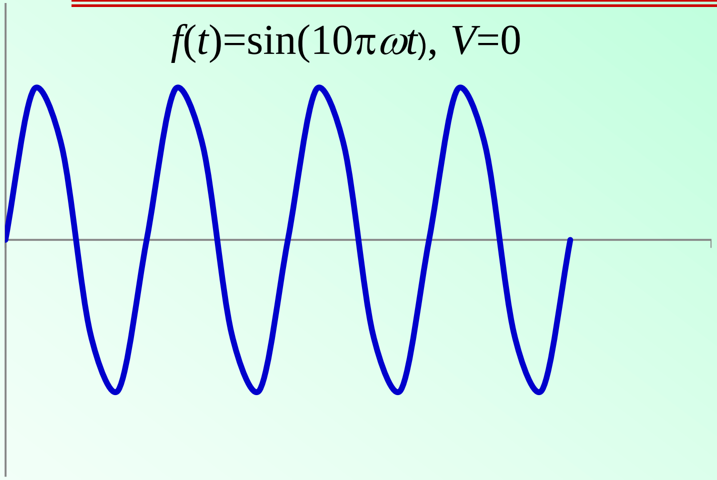


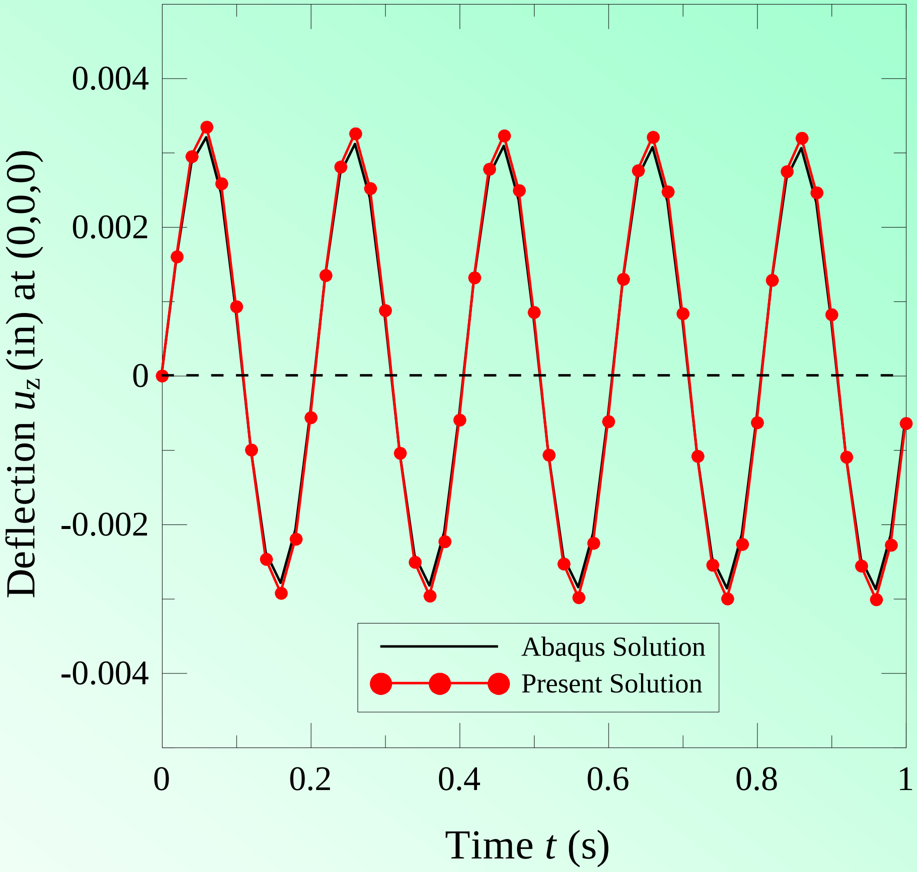


Abaqus CPU time: 50542 /49 s

Semianalytical Solution: <10 s

Verification-Dynamic Load

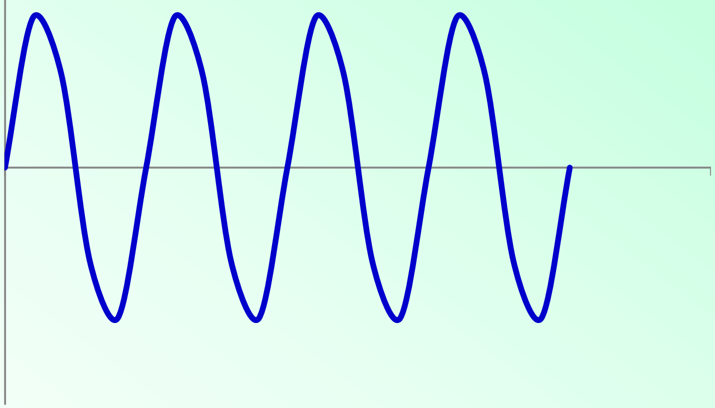




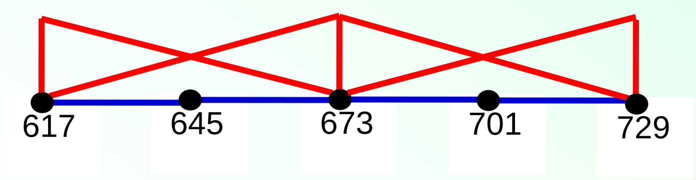
Abaqus CPU time: 88607/51 s
Semianalytical Solution: <1 s

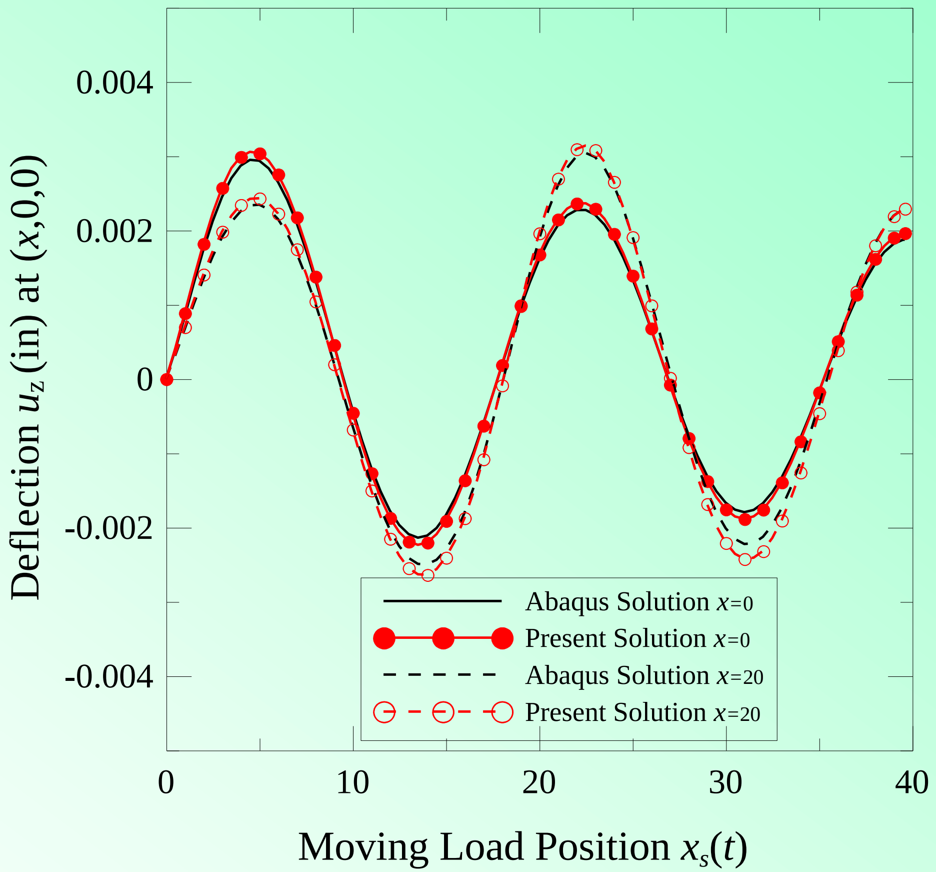
Verification-Moving Dynamic Load

f(*t*)=sin(10π*ωt*), *V*=5mph



Moving Configuration II

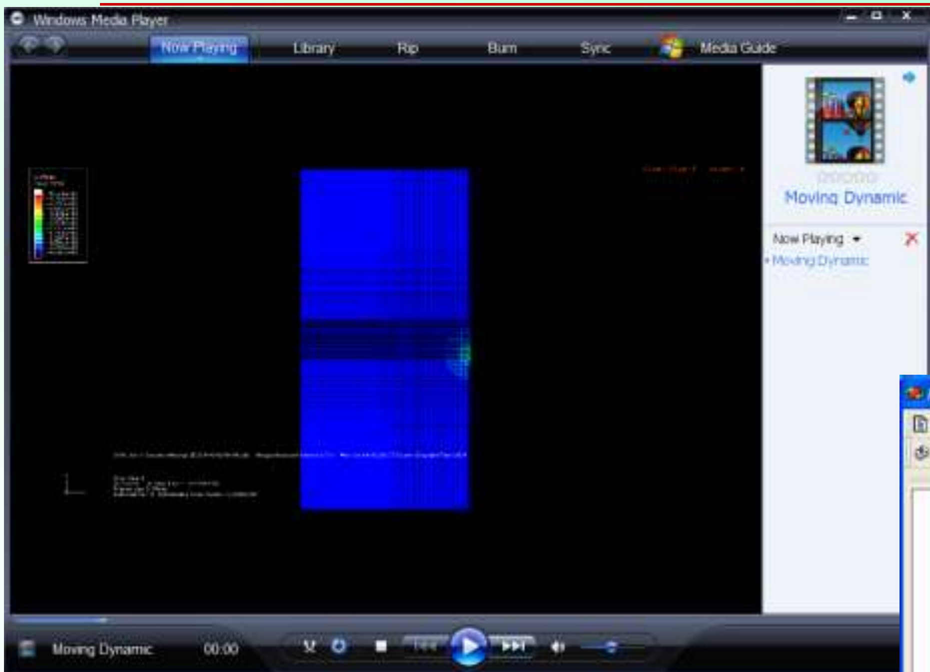


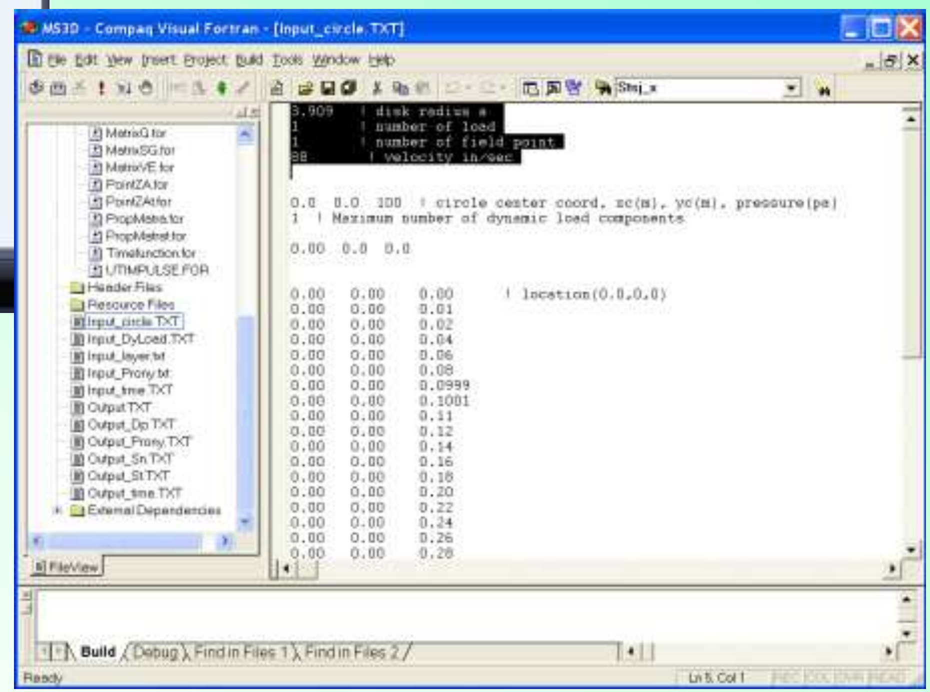


Abaqus CPU time: 87597/81 s

Semianalytical Solution: <10 s

Showcase

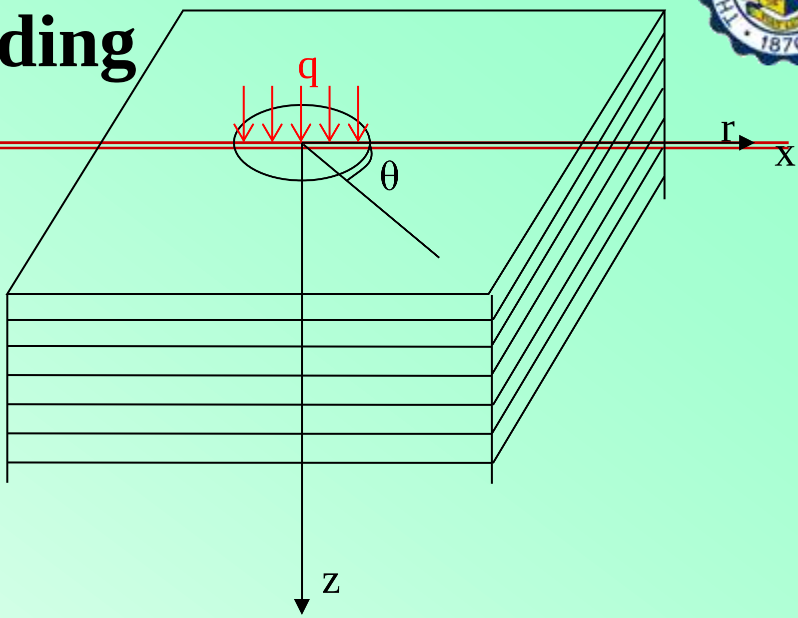




Abaqus CPU time: 87597/81 s

DELL OptiPlex 755
Intel Core 2 Quad CPU/Q6600 2.40GHz
RAM 3.25 GB

Boundary Conditions of Vertical Axis-Symmetric Circular Loading



$$\sigma_{zz} = \begin{cases} -q & r \le R \\ 0 & r > R \end{cases}$$

$$\sigma_{rz} = \sigma_{\theta z} = 0 \qquad r \ge 0$$

$$\mathbf{P} = \sigma_{rz}(r,\theta,0)\mathbf{e}_r + \sigma_{\theta z}(r,\theta,0)\mathbf{e}_\theta + \sigma_{zz}(r,\theta,0)\mathbf{e}_z = \sigma_{zz}(r,\theta,0)\mathbf{e}_z$$

$$\mathbf{P}\ (r,\theta,z_0) = \sum_m \int_0^\infty P_L(\lambda,m)\mathbf{L}(r,\theta) + P_M(\lambda,m)\mathbf{M}(r,\theta) + P_N(\lambda,m)\mathbf{N}(r,\theta)]\lambda d\lambda$$

$$P_L = -q\sqrt{2\pi}\int_0^R J_0(\lambda r) r dr = -\frac{\sqrt{2\pi}qRJ_1(\lambda r)}{\lambda} \qquad P_M = P_N = 0$$

Final Propagation Relation

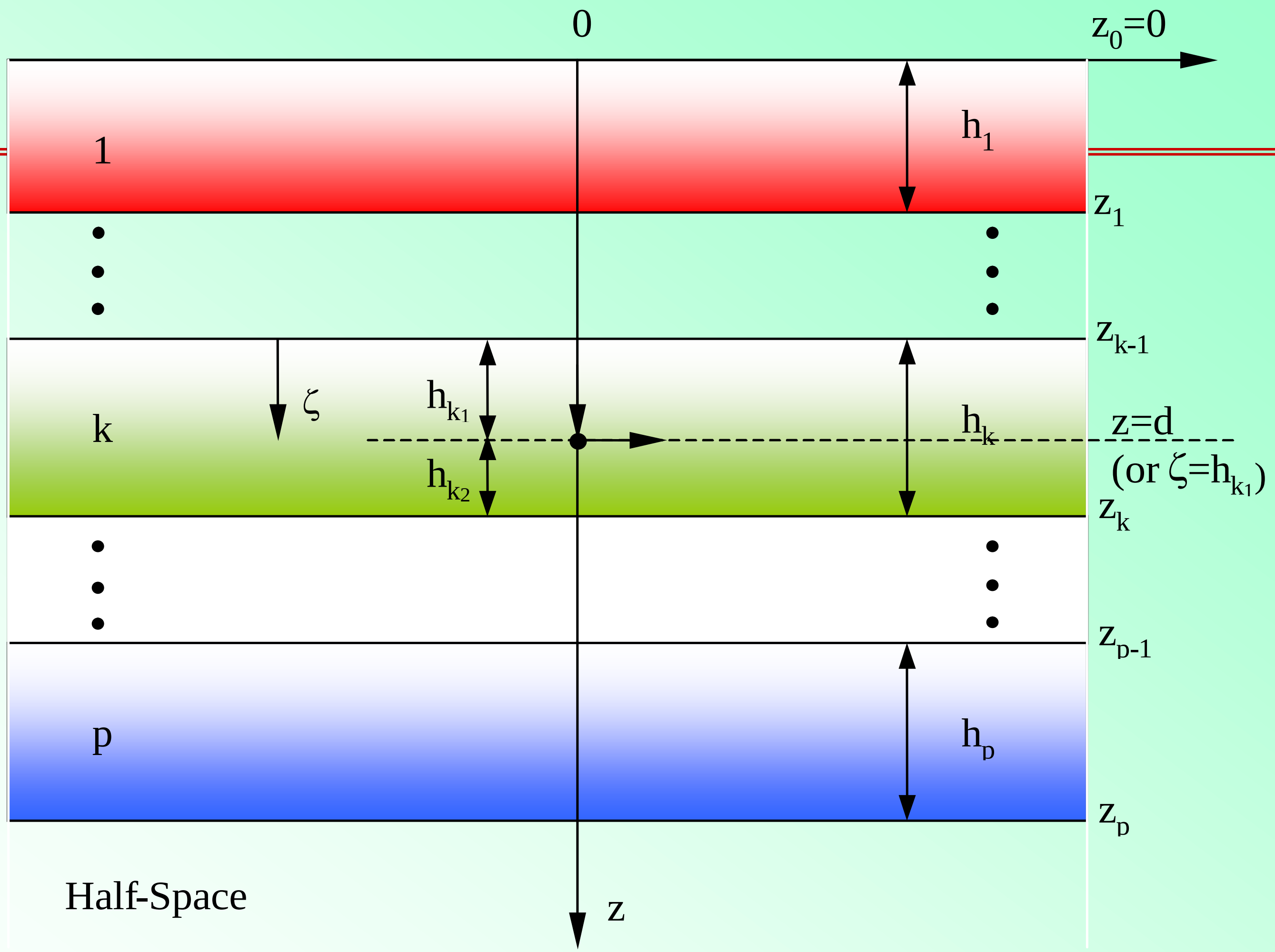
$$[E(z_0)] = \begin{bmatrix} U_L \\ \lambda U_M \\ T_L/\lambda \\ T_M \end{bmatrix}_{z=z_0} = [a_1][a_2]\cdots[a_p][E(z_p)] = [G]\begin{bmatrix} U_L \\ \lambda U_M \\ T_L/\lambda \\ T_M \end{bmatrix}_{z=z_N}$$

$$U_L(z_0) = ?, \qquad T_L(z_0)/\lambda = P_L/\lambda = -\frac{\sqrt{2\pi}qRJ_1(\lambda r)}{\lambda^2}$$

$$\lambda U_M(z_0) = ?, \qquad T_M(z_0) = P_M = 0$$

Solving for the unknown displacements on the surface z_0

Solution at any field point z



$$[E(z)] = [a_k(z - z_{k+1})][a_{k+1}]\cdots[a_N][E(z_N)]$$

Outline

--- Introduction of MultiSmart3D

--- Vertical Loading (details)

--- Horizontal Loading (details)

--- Other Methods and Comparisons

--- Effect of Transverse Isotropy

--- Moduli Backcalculation

--- Forward/Inverse with Bonding

--- Conclusions

Outline

1. Forward Calculations (MultiSmart3D)
--- General Formulation and Solutions
--- Graded Moduli
--- Transverse Isotropy/Shearing
--- Viscoelasticity
2. Inverse Calculations (BackGenetic3D)
--- Genetic Algorithm
--- Implementation
--- Preliminary Results
3. Conclusions