A variational inequality approach for lower bound limit analysis

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Outline of topics

1. Introduction
2. lower bound limit analysis
3. Variational Inequality Formulation
4. projection on Mohr–Coulomb cone
5. Examples

1. Introduction

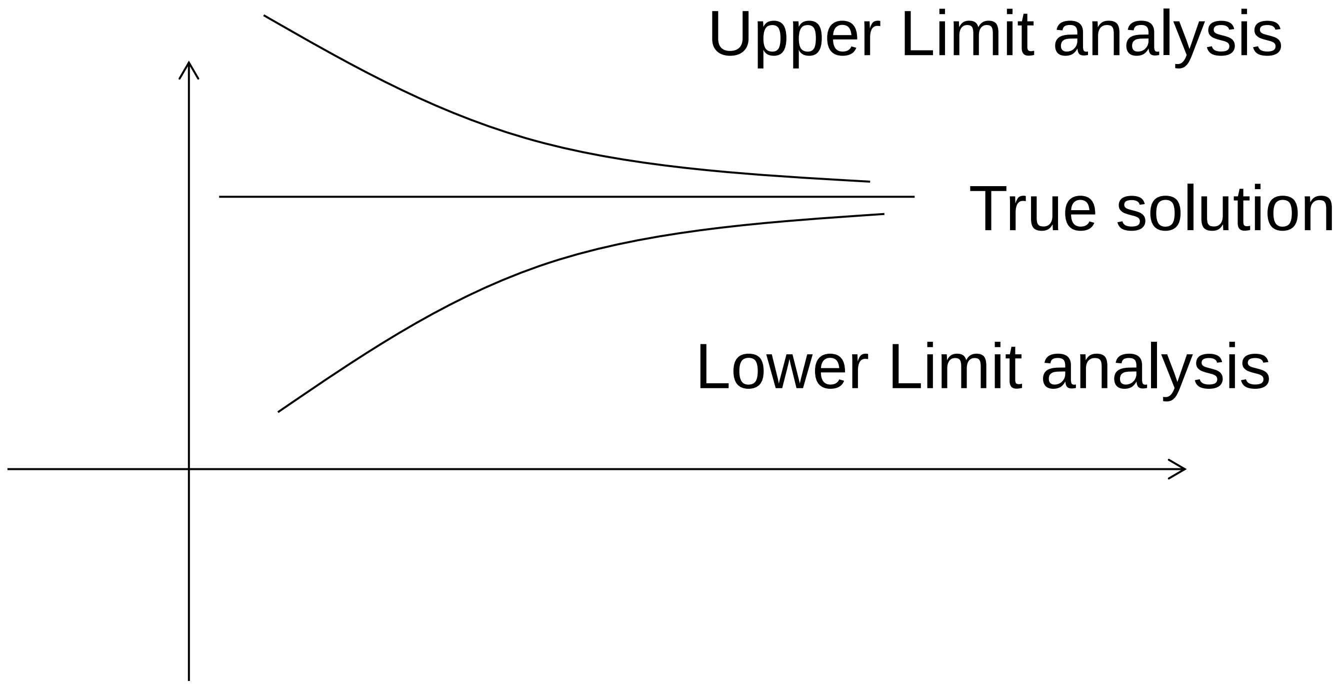
- Deformation problem and Stability problem

1. Introduction

- limit equilibrium method
- Continuum deformation analysis
- Limit analysis

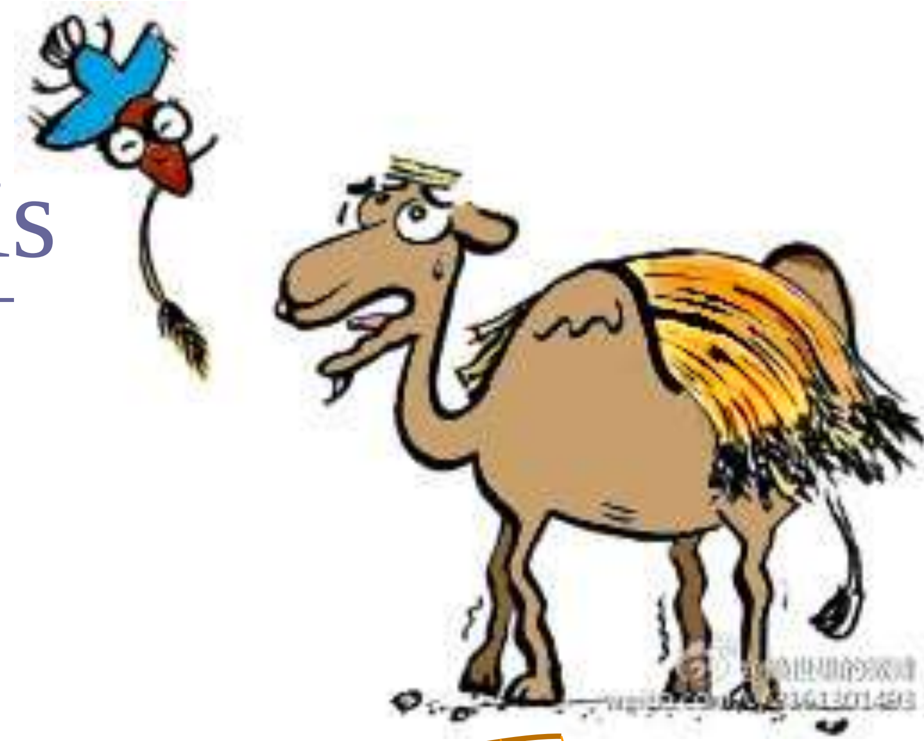
1. Introduction

- Limit analysis



2. Lower limit analysis

- Lower limit analysis

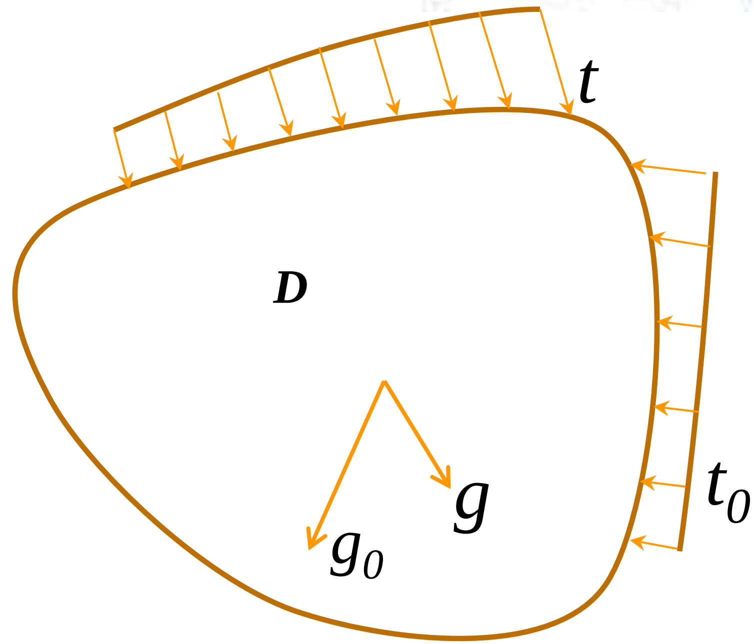


$$\max \eta$$

$$\sigma_{ij} n_j = t_{0i} \quad \sigma_{ij} n_j = \eta t_i$$

$$\sigma_{ij,j} + \eta g_i + g_{0i} = 0$$

$$f(\sigma_{ij}) \le 0$$



2. Lower limit analysis

- Lower limit analysis

$$\max\left\{\eta\left|\begin{array}{l}\sigma_{ij,j}+\eta g_i+g_{0i}=0,\\ \sigma_{ij}n_j=t_i(\mathrm{on}\,\partial D_{t0}),\\ \sigma_{ij}n_j=\eta t_i(\mathrm{on}\,\partial D_t),\sigma_{ij}\in\Omega\end{array}\right.\right\}$$

$$\Omega=\{\sigma_{ij}\,|\,f(\sigma_{ij})\le 0\}$$

2. Lower limit analysis

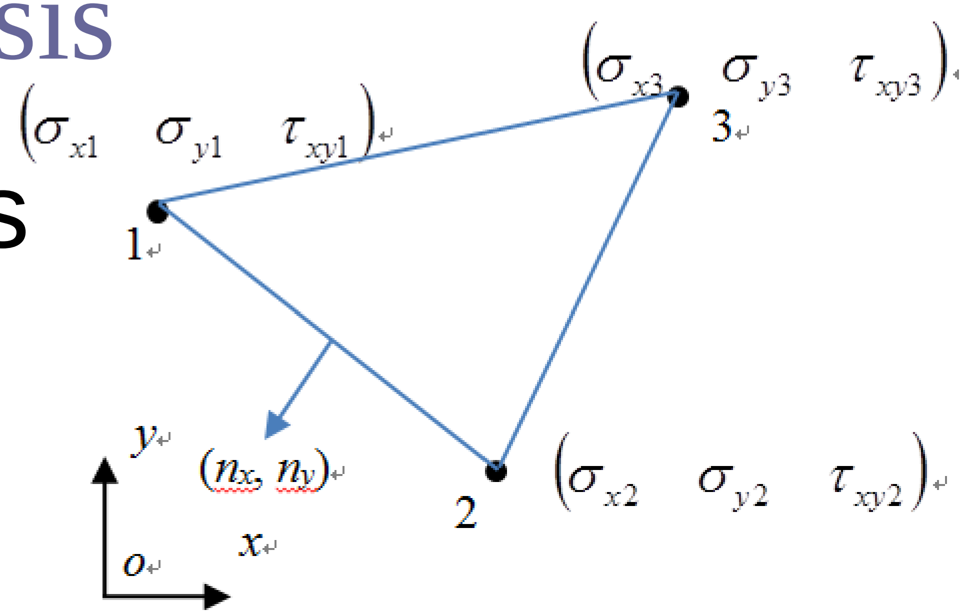
- Lower limit analysis

$$\sigma_x = \sum_{i=1}^{3} N_i \sigma_{xi}$$

$$\sigma_y = \sum_{i=1}^{3} N_i \sigma_{yi}$$

$$\tau_{xy} = \sum_{i=1}^{3} N_i \tau_{xyi}$$

$$\begin{cases} (\sigma_{ij}^1 - \sigma_{ij}^2) n_j = 0 \\ (\sigma_{ij}^3 - \sigma_{ij}^4) n_j = 0 \end{cases}$$

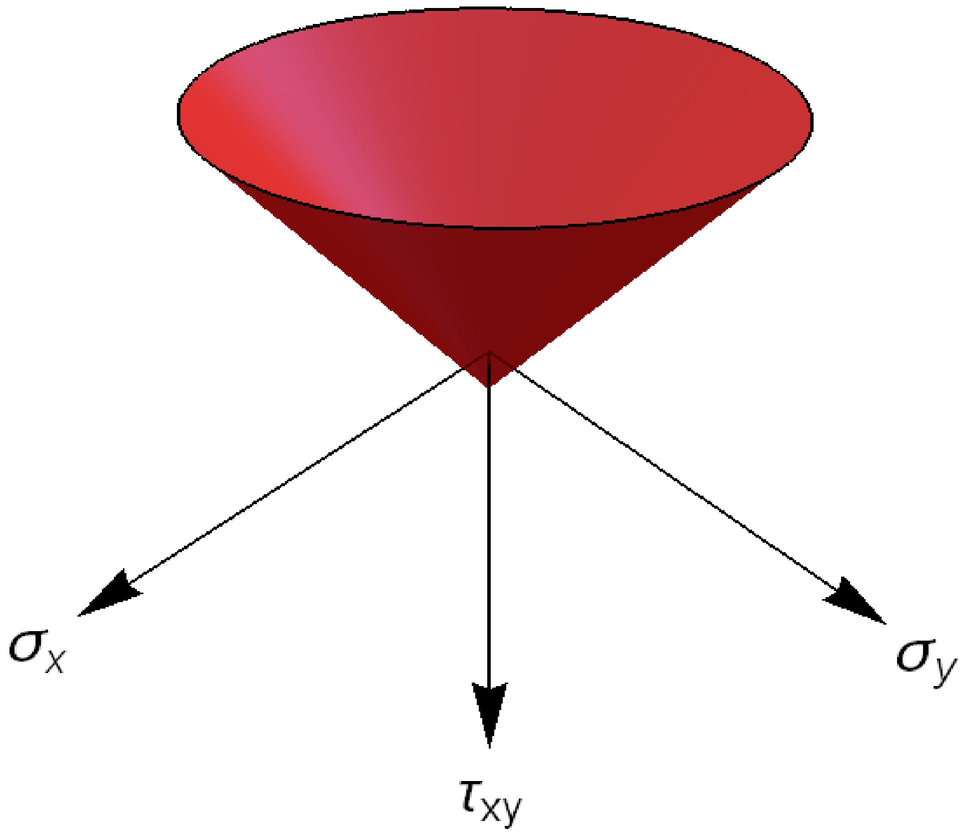


$$\max\{\eta | A_1 \sigma = \eta b_1 + b, \sigma \in \Omega\}$$

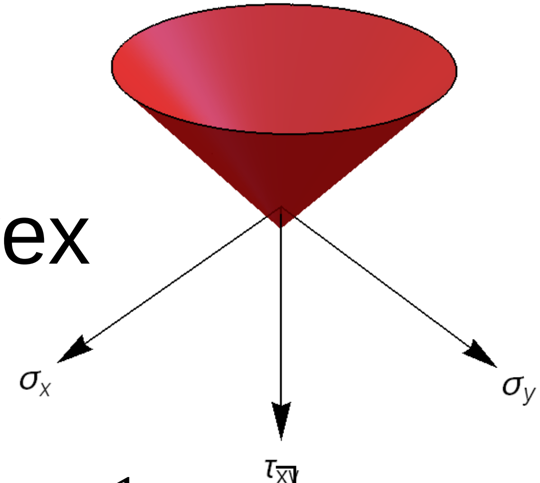
$$\Omega = \{\sigma | f_i(\sigma) \le 0, i = 1,2,\cdots n\}$$

2. Lower limit analysis

$$f(\sigma_x, \sigma_y, \tau_{xy}) = \sqrt{(\sigma_x - \sigma_y)^2 + (2\tau_{xy})^2} + (\sigma_x + \sigma_y)\sin\varphi - 2c\cos\varphi \leq 0$$



2. Lower limit analysis



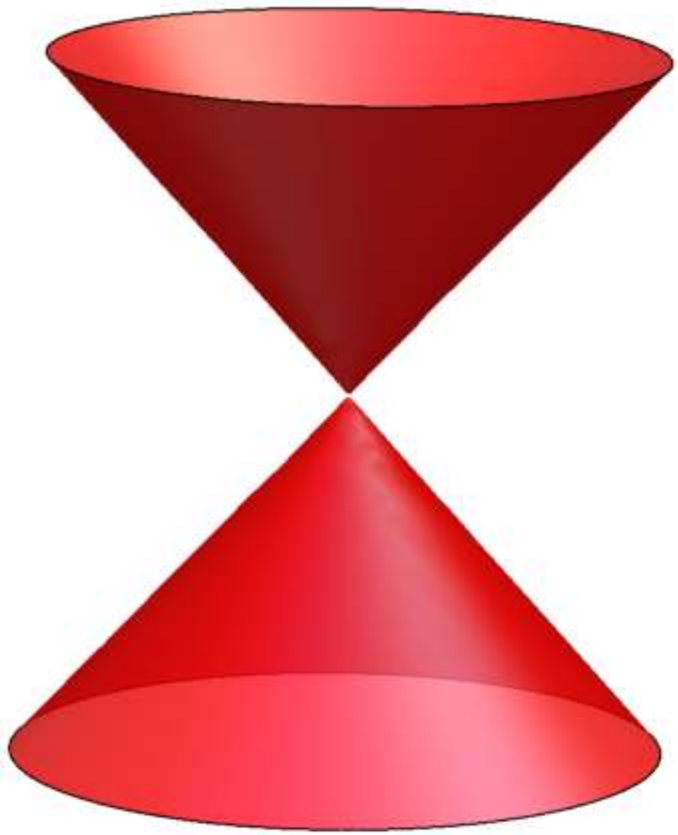
$$\mathbf{H} = \frac{4\tau_{xy}^{\ 2}}{\left[\left(\sigma_x - \sigma_y\right)^2 + \left(2\tau_{xy}\right)^2\right]^{3/2}} \begin{bmatrix} 1 & -1 & -\alpha \\ -1 & 1 & \alpha \\ -\alpha & \alpha & \alpha^2 \end{bmatrix} \qquad \alpha = \frac{\sigma_x - \sigma_y}{\tau_{xy}}$$

eigenvlues of H are 0(repeted) and

$$4\left[(\sigma_x - \sigma_y)^2 + 2\tau_{xy}^{\ 2}\right] \Big/ \left[(\sigma_x - \sigma_y)^2 + (2\tau_{xy})^2\right]^{3/2}$$

2. Lower limit analysis

$$f(\sigma_x, \sigma_y, \tau_{xy}) = (\sigma_x - \sigma_y)^2 + (2\tau_{xy})^2 - [(\sigma_x + \sigma_y)\sin\varphi - 2c\cos\varphi]^2 \le 0$$



3. Variational Inequality Formulation

- Lower limit analysis

Optimization Problem

$$v = \begin{pmatrix} \sigma \\ \eta \end{pmatrix} \qquad \min : \left\{ w^T v \,\middle|\, Av = b, v \in \Omega \right\}$$

Lagrangian Function

$$L(v, \lambda) = w^T v - \lambda^T (Av - b)$$

$$\max_{\lambda \in R^n} \min_{v \in \Omega} L(v, \lambda)$$

Saddle Point Problem

3. Variational Inequality Formulation

Lagrange Function $L(v,\lambda) = w^T v - \lambda^T (Av - b)$

a saddle point (v^*,λ^*) $\max\limits_{\lambda \in R^n} \min\limits_{v \in \Omega} L(v,\lambda)$

$$L(v^*,\lambda) \le L(v^*,\lambda^*) \le L(v,\lambda^*)$$

$$\left(\lambda - \lambda^*\right)^T \frac{\partial L(v,\lambda^*)}{\partial v}\bigg|_{v=v^*} \le 0 \qquad \left(v - v^*\right)^T \frac{\partial L(v,\lambda^*)}{\partial v}\bigg|_{v=v^*} \ge 0$$

3. Variational Inequality Formulation

$$L(v,\lambda) = w^T v - \lambda^T (Av - b)$$

$$\left(\lambda - \lambda^*\right)^T \frac{\partial L(v,\lambda^*)}{\partial v}\bigg|_{v=v^*} \le 0 \quad \left(v - v^*\right)^T \frac{\partial L(v,\lambda^*)}{\partial v}\bigg|_{v=v^*} \ge 0$$

$$u = \begin{pmatrix} v \\ \lambda \end{pmatrix}, F(u) = \begin{pmatrix} \nabla_v L(v,\lambda) \\ -\nabla_\lambda L(v,\lambda) \end{pmatrix} = \begin{pmatrix} w - A^T\lambda \\ Av - b \end{pmatrix}$$

$$u^* \in \Omega, \left(u - u^*\right)^T F\left(u^*\right) \ge 0, \forall u \in \Omega$$

3. Variational Inequality Formulation

$$u^* \in \Omega, \left(u - u^*\right)^T F\left(u^*\right) \geq 0, \forall u \in \Omega$$

- linear variational inequality

$$u^* \in \Omega, \left(u - u^*\right)^T \left(Mu^* + q\right) \geq 0, \forall u \in \Omega$$

$$M = \begin{pmatrix} 0 & -A^T \\ A & 0 \end{pmatrix}, q = \begin{pmatrix} w \\ -b \end{pmatrix}$$

3. Variational Inequality Formulation

- B. S. He

Step 0. Set $\beta_0 = 1$, $r_0 \in (0,1)$, $u^0 \in \Omega$ and k=0.

Step 1. $\tilde{u}^k = P_\Omega[u^k - \beta_k F(u^k)]$, $r_k = \beta_k \dfrac{\|F(u^k) - F(\tilde{u}^k)\|}{\|u^k - \tilde{u}^k\|}$.

while $r_k > r_0$

$$\beta_k = \frac{2}{3}\beta_k \min\{1, \frac{1}{r_k}\}, \tilde{u}^k = P_\Omega[u^k - \beta_k F(u^k)], r_k = \beta_k \frac{\|F(u^k) - F(\tilde{u}^k)\|}{\|u^k - \tilde{u}^k\|}$$

end(while)

$$d(u^k, \tilde{u}^k) = (u^k - \tilde{u}^k) - \beta_k[F(u^k) - F(\tilde{u}^k)], \alpha_k = \frac{(u^k - \tilde{u}^k)^T d(u^k, \tilde{u}^k)}{\|d(u^k, \tilde{u}^k)\|^2},$$

$$u^{k+1} = u^k - \gamma \alpha_k d(u^k, \tilde{u}^k)$$

if $r_k \le \mu$, $\beta_k = (3/2)\beta_k$, **End(if)**

Step 2. $\beta_{k+1} = \beta_k$ and $k = k+1$, go to Step 1.

4. projection on Mohr–Coulomb cone

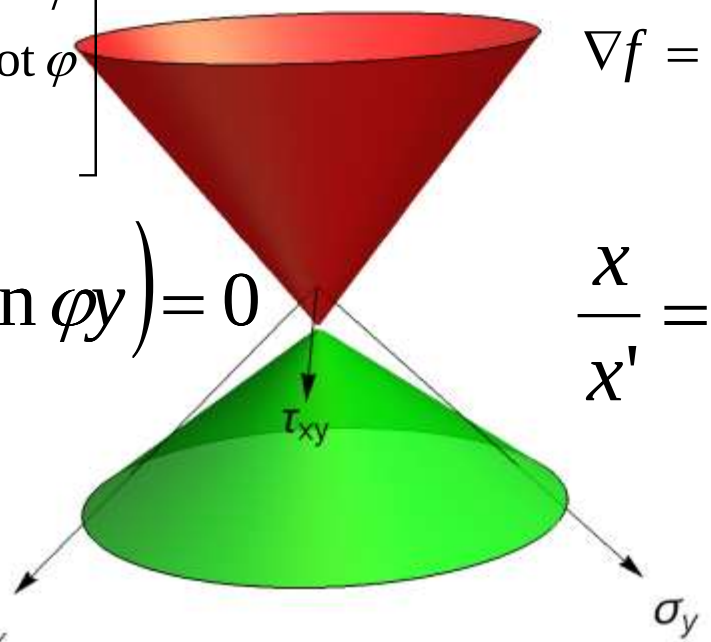
- normal cone of the Mohr-Coulomb cone at the apex

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_x - c\cot\varphi \\ \sigma_y - c\cot\varphi \\ \tau_{xy} \end{bmatrix}$$

$$\nabla f = \frac{2}{\sqrt{2x^2 + (2z)^2}} \begin{bmatrix} x \\ -\sin^2\varphi y \\ 2z \end{bmatrix}$$

$$f = \sqrt{2}\left(\sqrt{x^2 + 2z^2} + \sin\varphi y\right) = 0$$

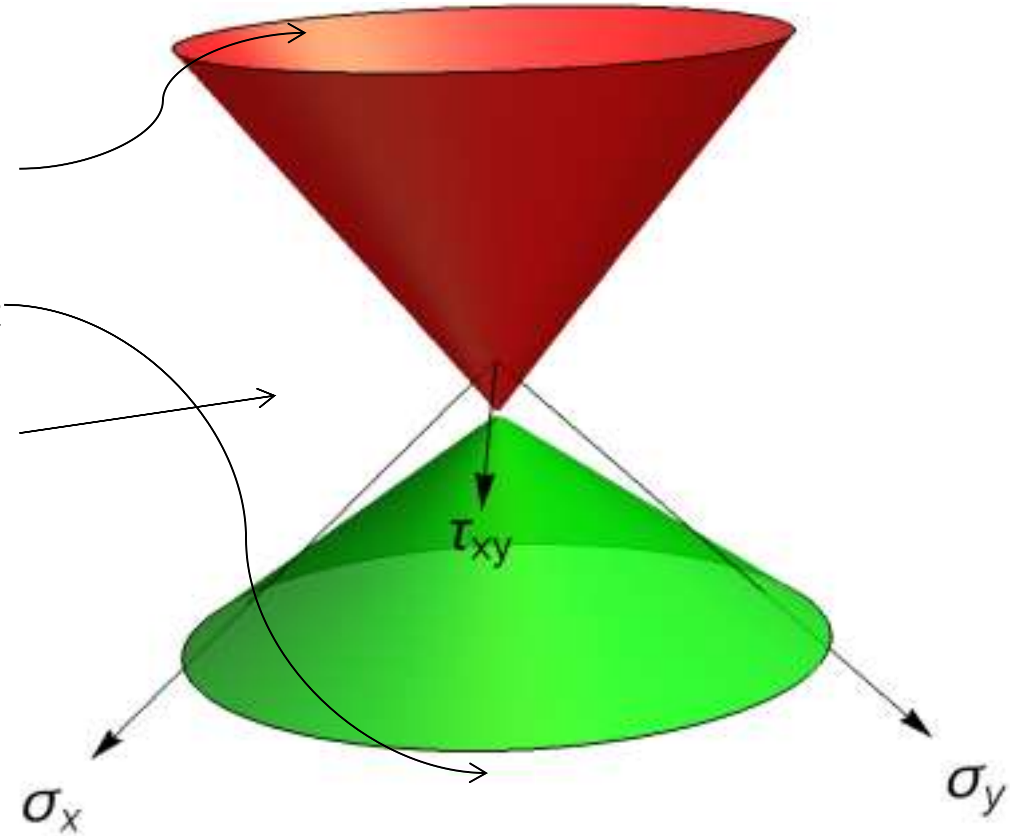
$$\frac{x}{x'} = \frac{-\sin^2\varphi y}{y'} = \frac{2z}{z'} = l$$



$$g(\sigma_x, \sigma_y, \tau_{xy}) = \sqrt{2}\left[\sqrt{(\sigma_x - \sigma_y)^2 + \tau_{xy}^{\ 2}} - (\sigma_x + \sigma_y)/\sin\varphi + 2c\cos\varphi/\sin^2\varphi\right] = 0$$

4. projection on Mohr–Coulomb cone

$$P_{\Omega}(\sigma)=\begin{cases}\sigma, & \sigma\in\Omega\\ \sigma_v, & \sigma\in\Omega_n\\ p_c, & others\end{cases}$$

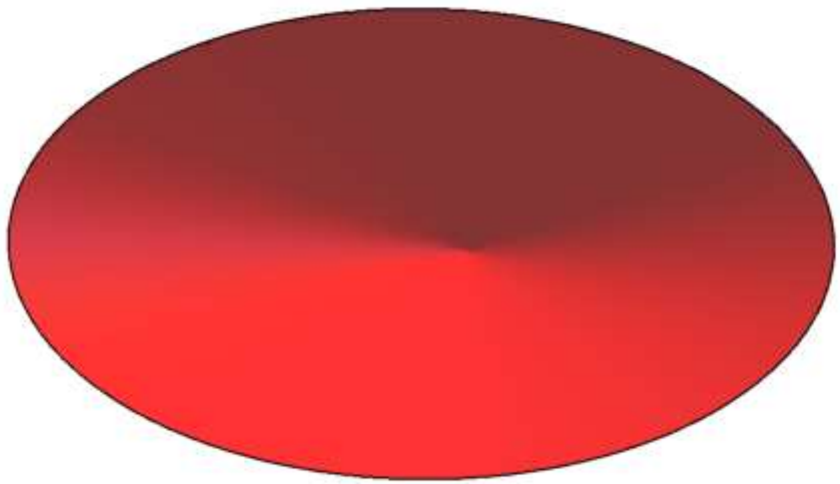


- How to solve p_c?

,

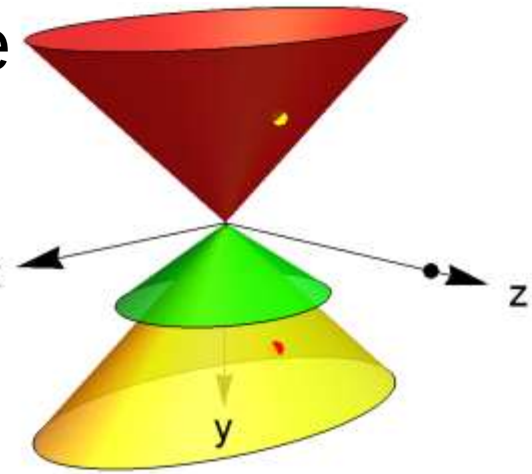
4. projection on Mohr–Coulomb cone

- Mohr-Couloumb cone is an *elliptical cone*



4. projection on Mohr–Coulomb cone

- Given (x_0,y_0,z_0),find (x,y,z) on the surface



$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s\begin{bmatrix} \cos\theta \\ -1/\sin\varphi \\ \sin\theta/\sqrt{2} \end{bmatrix}, s>0, -\frac{\pi}{2} \le \theta < \frac{\pi}{2}$$

$$\min : d^2 = (s\cos\theta - x_0)^2 + (s/\sin\varphi + y_0)^2 + \left(s\sin\theta/\sqrt{2} - z_0\right)^2$$

$$t = \tan\frac{\theta}{2} \qquad a_4 t^4 + a_3 t^3 + a_1 t - a_4 = 0$$

$$\Delta < 0$$

a pair of complex conjugate roots and *two real* roots, correspond to the projection point p_c(the yellow point) and a point (the red point) on the double cone.

5. Examples

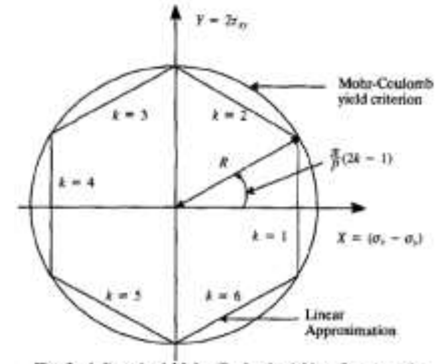
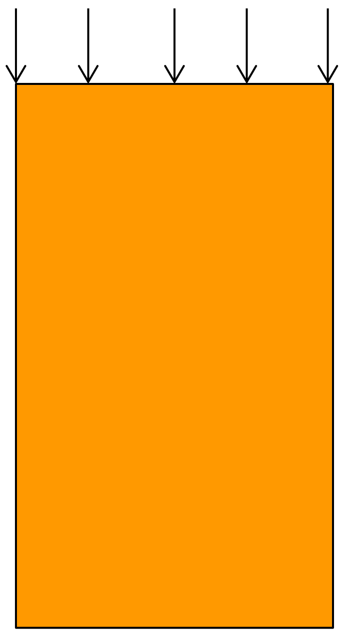


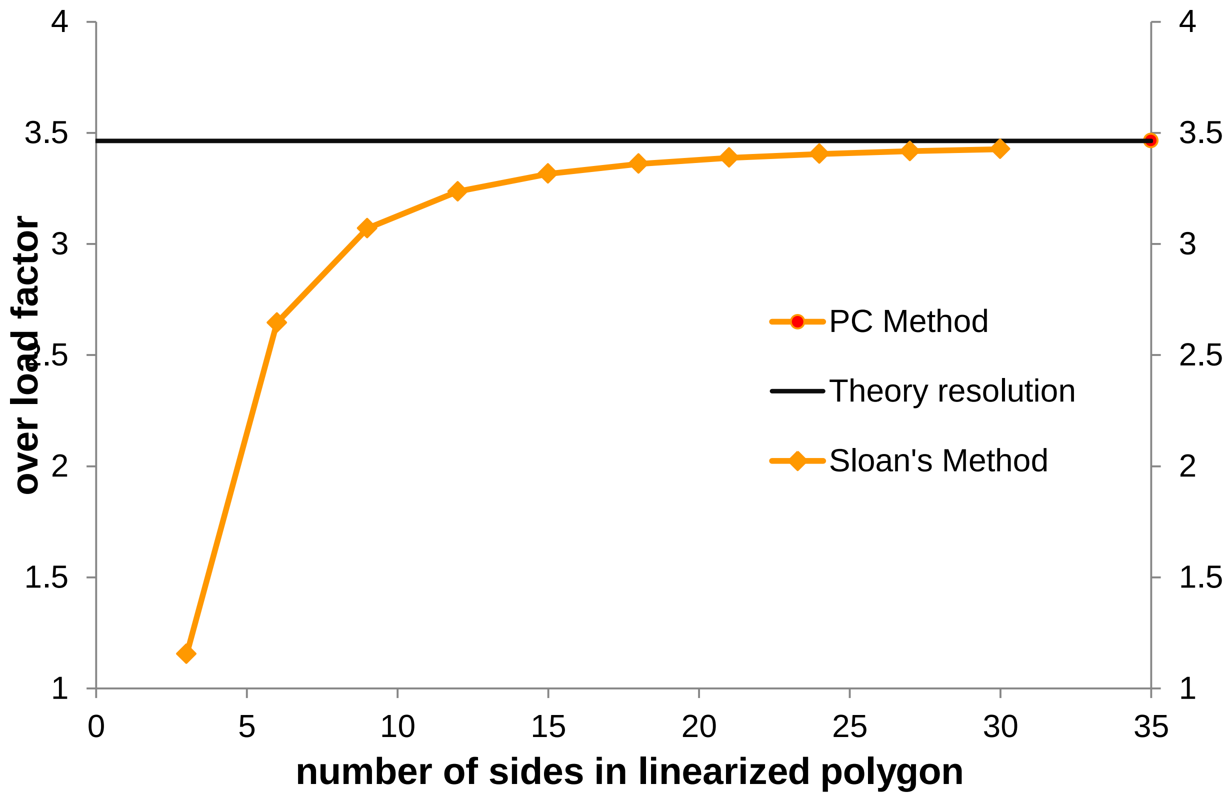
Fig. 3. A linearized Mohr–Coulomb yield surface ($p = 6$).

- uniaxial pressure

c=10000;φ=30°

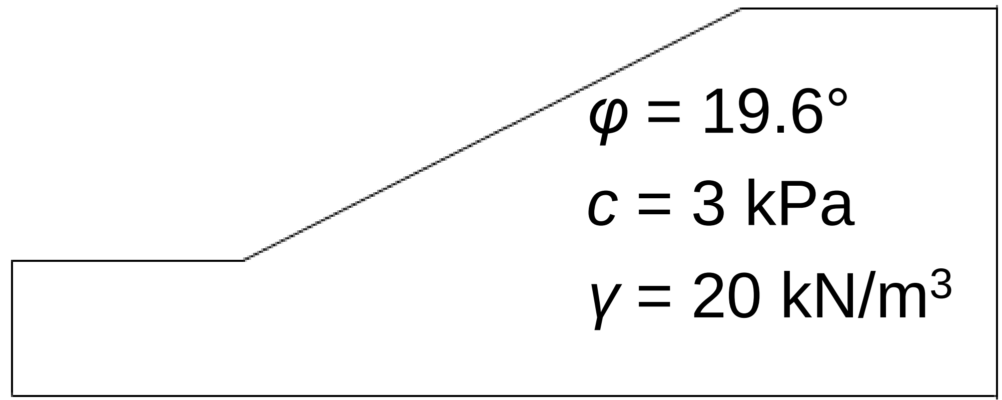
p=10000

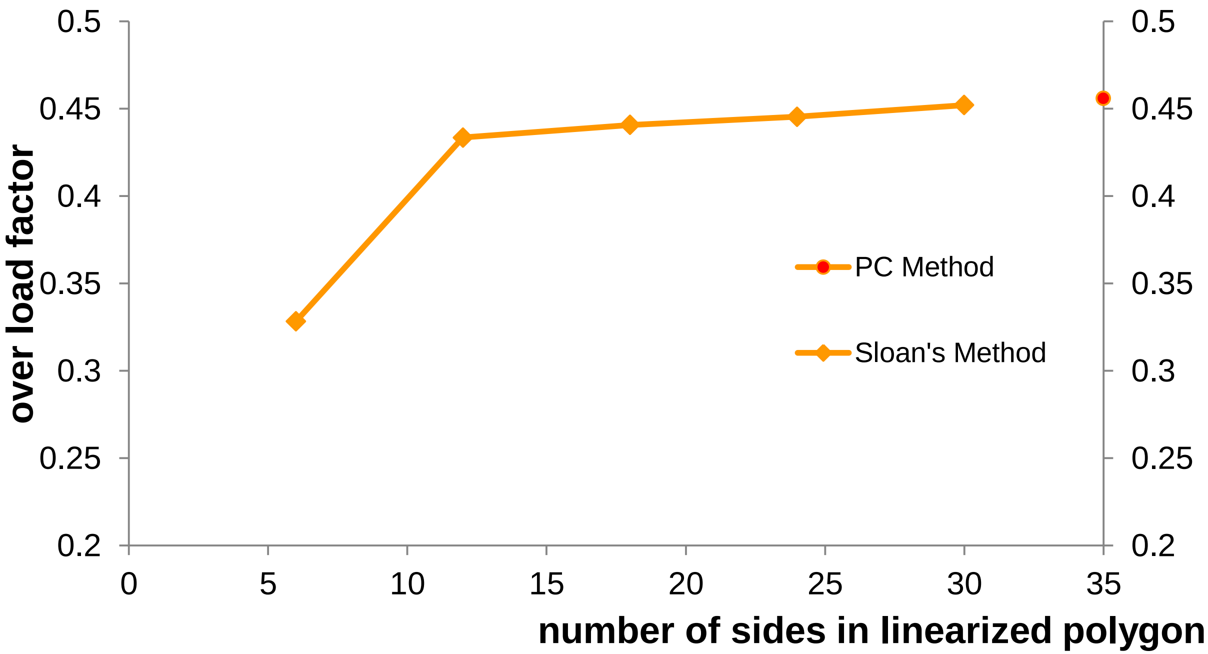




5. Examples

- Slope





Thank you all so much for your attention!

Does anyone have a question or comment?